A TREATISE

ON

ANALYTICAL GEOMETRY,

WITH

APPLICATIONS TO LINES AND SURFACES OF THE FIRST AND SECOND ORDERS.

BY

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PREFACE.

THE following Treatise on Analytical Geometry has been prepared for the use of the mathematical department in Columbia College and the School of Mines. Many excellent works on this branch of mathematics have been published, but the larger ones have proved too voluminous to be studied in the time allotted to the subject in colleges and schools of science, and the smaller ones are not sufficiently comprehensive to satisfy the growing wants of scientific education. The object of the present compilation is to deduce all the essential principles usually developed in the larger works, within the limits occupied by those of the smaller class.

The general plan of the work does not differ essentially from that adopted by the earlier writers on the subject, but in its execution, many changes have been introduced. The definitions have been revised, the explanations have been simplified, the demonstrations have been abbreviated, and every

branch has been illustrated by problems, intended to test the student's knowledge of the principles demonstrated. The method of treating tangents, normals, subtangents, and subnormals, has been much abridged, and, it is believed, correspondingly improved. Much care has been bestowed on the discussion of the general equation of the second degree, particularly with respect to the methods of testing the nature of the different curves that are represented by it.

The wood-cuts used in illustration have been kindly loaned by Prof. Davies. In thanking him for this act of courtesy, the writer begs leave also to acknowledge his indebtedness to him for other valuable aid. The continuous use of his text-books for more than a quarter of a century has not failed to leave an impress on the following pages.

COLUMBIA COLLEGE,
June 1, 1873.

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PART I.

ANALYTICAL GEOMETRY OF TWO DIMENSIONS.

I. Definitions and Introductory Remarks.

Definition of the Subject.

1. Analytical Geometry is a branch of Mathematics in which the properties and relations of lines and surfaces are investigated by the aid of Algebraic Analysis.

Method of Investigation.

2. The method of proceeding is as follows:

1°. All the points of the lines and surfaces to be considered are referred to fixed objects, by means of *elements*, called *co-ordinates.*

2°. The *relations* between these co-ordinates are expressed by *equations*, called *the equations of the lines and surfaces.*

3°. The properties and relations of the lines and surfaces are then deduced from these equations, by discussion and interpretation.

This method of investigation is often called *the method of co-ordinates.*

Division of the Subject.

3. Analytical Geometry may be divided into two parts: *Analytical Geometry of two dimensions,* which treats of lines lying wholly in a single plane, and *Analytical Geometry of three dimensions,* which treats of lines and surfaces situated in any manner in space.

In the former, *two* co-ordinates are sufficient to determine the position of a point; in the latter, *three* co-ordinates are necessary.

Systems of Co-ordinates in a Plane.

4. Any point of a plane is determined in position, if we know its distances from any two straight lines of the plane, that intersect each other. A point is also determined in position if we know its direction and its distance from a fixed point of the plane.

Both these methods of determining the positions of points are in common use, and each forms the basis of a *system of co-ordinates.* The system in which points are referred to straight lines is called the *rectilinear system,* and that in which they are referred to a fixed point is called the *polar system.*

Rectilinear System.

5. Let X′X and Y′Y be two straight lines intersecting at A, and let P be any point in their plane. Draw PO parallel to YA. The point P is given in position if we know AO and OP.

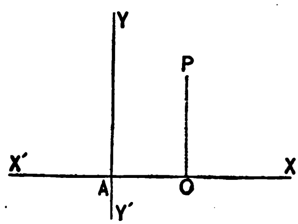

FIG. 1.

The lines X'X and Y'Y are called *co-ordinate axes,* and the point A is called the origin of co-ordinates.

The line X'X is called *the axis of abscissas,* or, *the axis of x.* It may have any direction, but it is usually assumed to be horizontal.

The line Y'Y is called *the axis of ordinates,* or, *the axis of y.* It may make any angle with the axis of x, but it is generally taken perpendicular to it. If the axes are perpendicular to each other, the system is said to be *rectangular,* otherwise it is *oblique.*

The distances OP and AO are rectilinear co-ordinates of P; OP is the *ordinate,* and AO the *abscissa.*

The *ordinate* of a point is its distance from the axis of x, measured on a line parallel to the axis of y. The point O, from which the ordinate springs, is called its *foot,* and the point P, where it terminates, is called its *extremity.*

It is agreed to consider distances estimated *upward* as *positive; consequently,* distances estimated downward are to be regarded as negative. Hence, if a point P is *above* the axis of x, its ordinate is *positive;* if *below* the axis of x, its ordinate is *negative;* and if on the axis of x, its ordinate is 0.

The *abscissa* of a point is the distance from the origin of co-ordinates to the foot of the ordinate. The point A, from which it springs, is the *origin;* and the point O, at which it terminates, is its *extremity.*

It is agreed to consider distances *toward the right* as positive; *consequently,* distances estimated *to the left* are to be regarded as *negative.* Hence, if P is *to the right* of the axis of y, its abscissa is *positive;* if it is to the

left of the axis of y, its abscissa is *negative;* and if it is on the axis of y, its abscissa is 0.

Of two ordinates, or of two abscissas, that is the *greater* whose *extremity* is nearer to $+\infty$, and that is the *less* whose *extremity* is nearer to $-\infty$.

Construction of Points.

6. If the abscissa of any point be denoted by x, and its ordinate by y, the point itself may be represented by the expression (x, y); thus, the expression $(2, 3)$ represents the *point whose abscissa is* 2, *and whose ordinate is* 3. By giving proper values to x and y, the expression (x, y) may be made to represent any point in the plane of the co-ordinate axes. If the values of x and y are known, the corresponding point is said to be *given,* and its position may be constructed as follows:

Lay off from the origin, on the axis of x, a distance equal to the given abscissa, to the right when the abscissa is positive, and to the left when it is negative; from the extremity of this distance lay off on a line parallel to the axis of y, a distance equal to the given ordinate, upward when the ordinate is positive and downward when it is negative; the extremity of the last distance is the position of the given point.

EXAMPLES.

1. Construct the point,

$$(2, 3).$$

Solution.—Lay off AO to the right equal to *two units* of a given scale; on a line parallel to AY lay off OP

upward and equal *three* parts of the same scale; P is the required point.

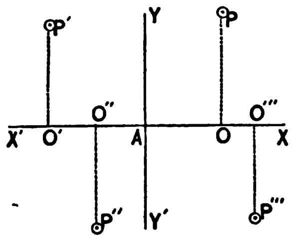

FIG. 2.

2. Construct the point,

$$(-3,\ 2).$$

Solution.—Lay off AO′ to the left, equal to three units; then lay off C′P′ upward, equal to two units; P′ is the required point.

3. Construct the point,

$$(-1\tfrac{1}{2},\ -3).$$

Solution.—Lay off AO″ equal to $1\frac{1}{2}$ units to the left and O″P″ equal to 3 units downward; P″ is the required point.

4. Construct the point,

$$(3,\ -2).$$

Solution.—Lay off AO‴ equal to three units to the right, and O‴P‴ equal to two units downward: P‴ is the required point.

5. Construct the straight line whose extremities are,

$$(3,\ -2) \quad \text{and} \quad (4,\ 2).$$

6. Construct the triangle whose vertices are,

$$(-4,\ 3),\quad (-2,\ 1),\quad \text{and} \quad (3,\ 5).$$

Polar System.

7. Let A be a fixed point of a plane, AX a fixed line and P any point of the plane. Draw AP. The point P is given in position if we know the angle XAP and the distance AP.

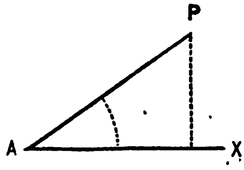

FIG. 3.

The line AX is called the *initial line*, and the point A is called the *pole*. The initial line may have any position in the plane, but it is usually drawn through A horizontally to the right.

The angle XAP and the distance AP are the polar co-ordinates of P. The former is called the *direction-angle*, and the latter is called the *radius-vector*.

The direction-angle is estimated from the initial line around to the radius-vector, as in trigonometry; the radius-vector is estimated from the pole outward. The direction-angle is usually positive, but the radius-vector may be either positive, or negative. In the latter case, the position of the point is on the radius-vector, prolonged backward through the pole.

Construction of Points in the Polar System.

8. If the direction-angle of any point be denoted by ϕ, and its radius-vector by r, the point itself may be represented by the expression (ϕ, r). By giving suitable values to ϕ and r, the expression (ϕ, r) may be made to represent any point in the plane. If the values of ϕ and r are known, the corresponding point is said to be *given*, and its position may be constructed as follows:

Draw a line from the pole, making an angle with the

initial line equal to the value of ϕ; on this, lay off from the pole, a distance equal to the value of r; the extremity of this distance is the required point.

EXAMPLES.

1. Construct the point,

$$(45^\circ, 4).$$

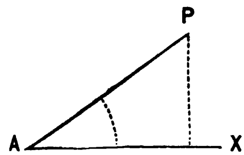

FIG. 4.

Solution.—From A draw a line making the angle XAP equal to 45°; lay off AP equal to four units of a given scale; P will be the required point.

2. Construct the points,

$$(180^\circ, 3) \quad \text{and} \quad (90^\circ, 6).$$

II. RELATIVE POSITION OF TWO POINTS IN A PLANE.

Proposition 1.—*To deduce a formula for the distance between two points, in terms of their rectangular co-ordinates.*

9. Let P and Q be the points. Draw their ordinates, and through P draw PC parallel to AX.

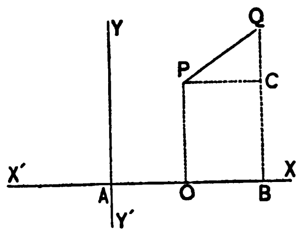

FIG. 5.

Denote the *first* point by (x', y'), the *second* point by (x'', y''), and the distance between them by d.

From the figure we have,

$$\overline{PQ}^2 = \overline{PC}^2 + \overline{CQ}^2;$$

But,

$$PQ = d, \quad PC = x'' - x', \quad \text{and} \quad CQ = y'' - y'.$$

Substituting these values, we have,

$$d^2 = (x'' - x')^2 + (y'' - y')^2.$$

Hence,

$$d = \sqrt{(x'' - x')^2 + (y'' - y')^2} \quad . \quad . \quad . \quad [1]$$

Which is the required formula.

EXAMPLES.

1. Find the distance from the point $(-8, -2)$ to the point $(3, 7)$.

Ans. $d = \sqrt{(3+8)^2 + (7+2)^2} = 14.21.$

2. Find the distance from $(-3, -4)$ to $(3, 4)$.

Ans. $d = \sqrt{(6)^2 + (8)^2} = 10.$

3. Find the distance from $(4, 2)$ to $(8, -4)$.

Ans. 7.21.

4. Find the distance between the points $(-2, -4)$ and $(6, 7)$. *Ans.* 13.6.

5. Find the sides of the triangle whose vertices are $(1, 6)$, $(2, -3)$, and $(4, -6)$.

Ans. 9.06, 12.37, and 3.6.

The preceding results may be verified by constructing the given points, and then measuring the distances between them by a scale of equal parts.

Proposition 2.—*To find the inclination of the line joining two given points.*

10. The *inclination* of a line is the angle that it makes with the axis of x. The vertex of this angle is the point in which the line, or line produced, intersects the axis of x, and the angle itself is estimated from the positive direction of the axis of x around to the given line, as in trigonometry. The inclination may have any value from 0 to 180°.

Let P and Q be the two points, PQ a line joining them, and PC a line parallel to the axis of x. Then will BHQ, or its equal, CPQ, be the required inclination.

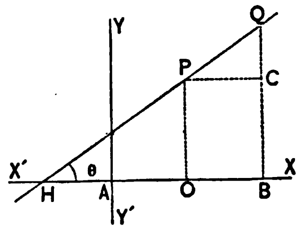

FIG. 6.

Denote the first point by (x', y'), the second point by (x'', y''), and the angle CPQ by θ.

From the figure, we have,

$$\tan\theta = \frac{CQ}{PC}.$$

But,

$$CQ = y'' - y' \quad \text{and} \quad PC = x'' - x'.$$

Substituting these values, we have,

$$\tan\theta = \frac{y'' - y'}{x'' - x'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad [2]$$

which is a formula for finding the tangent of the inclination.

The tangent of the inclination is called the *slope* of

the line. The word *slope*, as here employed, is nearly synonymous with the term *grade* in engineering.

In applying formula [2], to find the slope of a line through two points, we always subtract the *ordinate* of the *first* point from that of the *second*, for the numerator, and the *abscissa* of the *first* point from that of the *second*, for the denominator.

Of two given points, that is the *first* which has the *smaller* abscissa (Art. **5**). Hence, the denominator of the slope is always positive. The slope itself is *positive* if $y'' > y'$, and *negative* if $y'' < y'$. In the former case, the line slopes upward in passing from the first point toward the second; and in the latter case, it slopes downward. When a line slopes upward, its inclination is less than 90°; when it slopes downward, its inclination is greater than 90°.

If the slope is known, the inclination can be found *directly*, from a table of natural tangents, or *indirectly*, from a table of logarithmic tangents. In the latter case, we add 10 to the logarithm of the slope (disregarding the sign), and take the corresponding angle from a table of logarithmic tangents. If the slope is *plus*, the angle thus found is *the required inclination*; if the slope is *minus*, the angle found is *the supplement of the inclination*.

EXAMPLES.

1°. Find the slope and inclination of a line joining the points,

$$(2, 3) \quad \text{and} \quad (5, -1).$$

Solution.—We have from equation [2],

$$\tan\theta = \frac{-1-3}{5-2} = -\frac{4}{3}.$$

$\therefore$ $\log\tan\theta = \log 4 - \log 3 + 10 = 10.124939.$

The corresponding angle is

$53^\circ\ 7'\ 48''$; $\therefore$ $\theta = 180^\circ - 53^\circ\ 7'\ 48'' = 126^\circ\ 52'\ 12''.$

2°. Find the slope and inclination of the line joining the points

$$(-3, -4) \quad \text{and} \quad (3, 4).$$

Ans. $\tan\theta = \frac{4}{3}$; and $\theta = 53^\circ\ 7'\ 48''.$

3°. Find the slope and the inclination of a line joining the points,

$$(-8, -2) \quad \text{and} \quad (3, 7),$$

to the nearest minute.

Ans. $\tan\theta = .818181$; and $\theta = 39^\circ\ 17'.$

4°. Find the slope and inclination of the line joining the points,

$$(1, 6) \quad \text{and} \quad (12, -3),$$

to the nearest minute.

Ans. $\tan\theta = -0.818181$; and $\theta = 140^\circ\ 43'.$

5°. Find the slope and inclination of the line passing through the points,

$$(1, 6) \quad \text{and} \quad (14, -6),$$

to the nearest minute.

Ans. $\tan\theta = -0.923077$; and $\theta = 137^\circ\ 17'.$

III. Of the Straight Line.

Proposition 3.—*To find the equation of a straight line referred to rectangular axes.*

11. The *equation of a line* is an equation that expresses the relation between the co-ordinates of every point of the line.

Let AX and AY be rectangular co-ordinate axes; HQ, any straight line in their plane; P, any point of that line; OP, the ordinate of P; and let ED be parallel to AX.

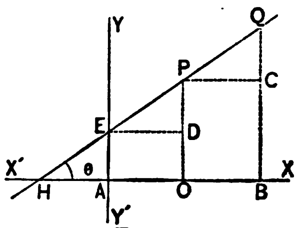

Fig. 7.

Denote the co-ordinates AO and OP by x and y; the distance AE by b; the inclination AHE, or its equal, DEP, by θ; and let the tangent of θ be denoted by a.

From the figure, we have,

$$OP = OD + DP.$$

But,

$$OP = y, \quad OD = AE = b, \quad \text{and} \quad DP = ED \tan\theta = ax.$$

Substituting these values, we have,

$$y = ax + b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [3]$$

But the point P is, by hypothesis, *any point* of the line HQ; hence, equation [3] expresses the relation between the co-ordinates of *every* point of HQ; it is, therefore, the equation of that line, *which was to be found.*

Definitions of Constants and Variables.

12. The equation of a line contains two kinds of quantities: *constants*, whose values do not change for the same line, and *variables*, which admit of all possible values that will satisfy the equation into which they enter. In equation [3], a and b are constants, and x and y are variables.

Constants are of two kinds: *absolute constants*, whose values are fixed; and *arbitrary constants*, to which we may assign any reasonable values. In equation [3], a and b are *arbitrary constants*.

The quantity b, which denotes the distance AE, is called *the intercept*; by giving it a suitable value we may make HQ *pass through any point* on the axis of y. The quantity a, is the *slope*; by giving it a suitable value, we may make HQ take *any direction* with respect to the axis of x.

If the constants that enter the equation of a line are *arbitrary*, the line is said to be *given in kind*, that is, it may be any line of a given class; if the constants are *absolute*, the line is said to be *completely given*, that is, the equation represents some particular line of a given class.

The variables represent the co-ordinates of each and every point of the line; that is, if x represents the abscissa of any point of the line, y represents the ordinate of the same point. This principle enables us to find the co-ordinates of different points of a line whose equation is given. To do this, *assume* any value for x; substitute it in the given equation and *deduce* the corresponding

value of y; the assumed and deduced values will be the co-ordinates of one point of the line. In like manner the co-ordinates of other points may be found.

It is customary to assume values for x and to deduce corresponding values for y; for this reason, x is called the *independent variable*, and y is then said to be a *function of* x. One quantity is a *function* of another when the two quantities are so connected that no change can take place in the latter, without producing a corresponding change in the former. The fact that two quantities are so connected, may be expressed as follows

$$y = f(x), \quad \text{or} \quad f(x, y) = 0.$$

The former equation shows that y is a function of x, and the latter that x and y depend on each other, without indicating which is the function, or which the independent variable.

Proposition 4.—*To show that every equation of the first degree between two variables is the equation of a straight line.*

13. Every equation of the first degree between two variables can be reduced to the form,

$$my + nx + p = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Solving with respect to y, we have,

$$y = -\frac{n}{m}x - \frac{p}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which is identical in form with equation [3], hence, it is the equation of a straight line; the equation from which

it was derived is, therefore, the equation of a straight line, *which was to be proved.*

The *intercept* of the line (2) is $-\frac{p}{m}$, and its *slope* is $-\frac{n}{m}$; hence, to find these elements, from the equation of a straight line, we solve the equation with respect to y; the coefficient of x in the resulting equation is the *slope*, and the absolute term is the *intercept.*

In equation (1) there are *apparently* three arbitrary constants, whereas there are *in reality* but two, as is shown in equation (2). Before judging of the number of arbitrary constants in an equation, let both members be divided by the coefficient of one of the terms; the number of constants remaining will be the true number. Every line can be made to fulfill as many reasonable conditions as its equation contains arbitrary constants.

Construction of a Straight Line.

14. Let it be required to construct the line whose equation is,

$$8y - 6x - 5 = 0.$$

Solving with respect to y, we have,

$$y = \frac{3}{4}x + \frac{5}{8}.$$

Here the intercept is $\frac{5}{8}$ and the slope $\frac{3}{4}$.

Lay off AE equal to $\frac{5}{8}$; through E draw ED parallel to AX, and make it equal to 1; through the ex-

tremity D, draw DP perpendicular to ED and make it equal to $\frac{3}{4}$; then will EP be the required line, because it has the same intercept and the same slope.

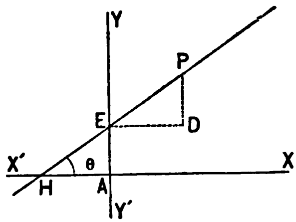

FIG. 8.

In like manner, other lines may be constructed.

If the intercept is *negative*, AE is to be laid off *downward;* if the slope is *negative*, DP is to be laid off *downward.*

EXAMPLES.

1°. Construct the line,

$$3x - 2y + 4 = 0.$$

2°. Construct the line,

$$2x - y - 3 = 0.$$

3°. Construct the line,

$$y + 2x - 4 = 0.$$

Lines may be constructed by points. To illustrate the method, let us resume the line whose equation is,

$$8y - 6x - 5 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Making $x = 1$, in equation (1), we find,

$$y = 1\tfrac{3}{8}.$$

Again, making $x = 2$, we find,

$$y = 2\tfrac{1}{8}.$$

Hence, the points,

$$(1, 1\tfrac{3}{8}) \quad \text{and} \quad (2, 2\tfrac{1}{8}),$$

are points of the given line.

Construct these points by the method of Article 6, and draw a straight line through them; this will be the required line.

In like manner, other lines may be constructed.

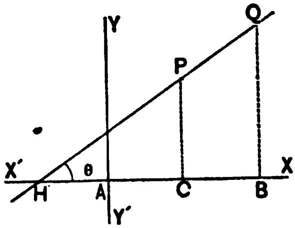

Fig. 9.

Let the following lines be constructed by points:

4°. $3x - 2y + 5 = 0.$

5° $2x - 5y - 1 = 0.$

6°. $3x + 2y + 5 = 0.$

7°. $-x + 3y - 4 = 0.$

To find the point in which a line cuts the axis of x, we make $y = 0$ in the equation of the line, and the corresponding value of x will be the abscissa of the point of intersection.

In like manner, if we make $x = 0$, and find the corresponding value of y, it will be the ordinate of the point in which the line cuts the axis of y.

Making $y = 0$, in equation (1), we find,

$$x = -\frac{5}{6}.$$

Making $x = 0$, we find,

$$y = \frac{5}{8}.$$

Hence, that line cuts the axis of x at the point $(-\frac{5}{6}, 0)$, and the axis of y at the point $(0, \frac{5}{3})$.

These results may be used as *checks*, to test the accuracy of the preceding constructions; they may also be used as a means of constructing lines by points.

Proposition 5.—*To find the equation of a straight line passing through a given point.*

15. Let (x', y') be the given point; assume the equation of a straight line,

$$y = ax + b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [3]$$

If the given point is on this line, its co-ordinates must satisfy the equation of the line, giving,

$$y' = ax' + b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Conversely, if equation (1) is satisfied, the line [3] must pass through the given point. Introducing this condition by subtracting (1) from [3], we have,

$$y - y' = a(x - x') \quad . \quad . \quad . \quad . \quad . \quad [4]$$

which is the equation of a straight line, with the condition introduced that it shall pass through the given point; *it is, therefore, the required equation.*

Equation (1) is called an *equation of condition. An equation of condition* is an equation that must be satisfied in order that a given condition may be fulfilled.

To find the equation of condition that places a point on a line, we substitute the co-ordinates of the point for the variables, in the equation of the line; if the line is completely given, and the position of the point is *ar-*

bitrary, the resulting equation is *the equation of condition that places the point on the line;* if the line is *given in kind,* and the position of the point is *fixed,* the resulting equation is *the equation of condition that causes the line to pass through the point.*

It may be shown that equation [4] is the equation of a straight line passing through the point (x', y') as follows:

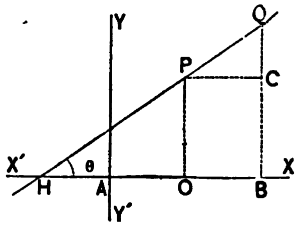

FIG. 10.

Let P be the given point (x', y'), and let Q be any point of the line HP, denoted by (x, y).

From the figure we have,

$$CQ = PC \tan CPQ.$$

But,

$$CQ = y - y', \quad PC = x - x', \quad \text{and } \tan CPQ = \tan \theta = a.$$

Substituting these values, we have,

$$y - y' = a(x - x'),$$

which is the same as equation [4].

The value of a in this equation is arbitrary, as it should be; for, an infinite number of straight lines can be drawn through (x', y').

Proposition 6.—*To find the equation of a straight line passing through two given points.*

16. Let (x', y') and (x'', y'') be the given points;

assume the equation of a straight line passing through the point (x', y'),

$$y - y' = a(x - x') \quad . \quad . \quad . \quad . \quad . \quad [4]$$

If we make a, in equation [4], equal to the slope of the line joining (x', y') and (x'', y''), (Art. **10**), the resulting equation will be the equation of a straight line passing through both the given points.

Making this substitution, we have,

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \quad . \quad . \quad . \quad . \quad [5]$$

which is the required equation.

EXAMPLES.

1. Required the equation of a straight line passing through the points,

$$(2, 3) \text{ and } (3, 2).$$

Ans. $y = -x + 5.$

2. Find the equation of a straight line passing through the points,

$$(-2, 3) \text{ and } (3, -2).$$

Ans. $y = -x + 1.$

3. Find the equation of a straight line passing through the points,

$$(4, -\frac{1}{2}) \text{ and } (2, -\frac{1}{4}).$$

Ans. $y = -\frac{x}{8}.$

Proposition 7.—*To find the intersection of two lines.*

17. If two lines intersect, the co-ordinates of their common point must satisfy the equations of both lines at the same time; that is, they must make the equations *simultaneous;* conversely, the values of x and y that make the two equations simultaneous, must be the co-ordinates of the point of intersection. Hence, to find the intersection of two lines, combine their equations and find the corresponding values of x and y; these will be the co-ordinates of the required point.

Let it be required to find the point of intersection of the straight lines, whose equations are as follows:

$$y = 2x - 5,$$
$$y = -\frac{1}{2}x + 5.$$

Combining these equations, we find,

$$x = 4 \quad \text{and} \quad y = 3;$$

hence the two lines intersect in the point,

$$(4, 3).$$

These values of x and y may be verified by substituting them in the given equations. Making the substitutions, we see that both equations are satisfied.

EXAMPLES.

1°. To find the intersection of the following lines,

$$3x + 7y = 47,$$

and,

$$8x - y = 27. \qquad \textit{Ans. } (4, 5).$$

2°. Find the intersection of the lines,

$$3x + 2y - 12 = 0,$$

and,

$$4x + 3y - 17 = 0.$$

Ans. (2, 3).

3°. Find the intersection of the lines,

$$2x + 9y - 20 = 0,$$

and,

$$4x + y \quad - 6 \quad = 0.$$

Ans. (1, 2).

4°. Find the intersection of the lines,

$$\frac{x}{3} + 2y \quad - 5 = 0,$$

and,

$$\frac{2x - 1}{5} - y + 1 = 0.$$

Ans. (3, 2).

Let the above results be verified by construction.

Proposition 8.—*To deduce a formula for finding the angle between two straight lines.*

18. Let DP and BP be the two lines. Denote the inclination of DP by θ, the inclination of BP by θ', and the required angle BPD by a.

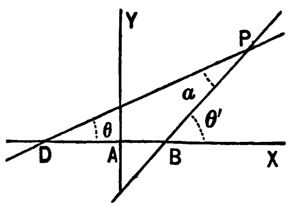

FIG. 11.

We have, (Legendre, B. I., Prop. 25, cor.),

$$a = \theta' - \theta.$$

Hence (Trig., Art. 65),

$$\tan \alpha = \frac{\tan \theta' - \tan \theta}{1 + \tan \theta \tan \theta'}.$$

But $\tan \theta$, and $\tan \theta'$, are the slopes of the given lines; denoting these by a and a' respectively, we have,

$$\tan \alpha = \frac{a' - a}{1 + a'a} \quad . \quad . \quad . \quad . \quad . \quad . \quad [6]$$

which is the required formula.

The two lines will be parallel if $\tan \alpha = 0$; that is, if,

$$a' = a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [7]$$

The two lines will be perpendicular to each other if,

$$\tan \alpha = \infty,$$

that is, if,

$$1 + a'a = 0,$$

or if,

$$a' = -\frac{1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [8]$$

Equation [7] is the equation of condition for parallel lines, and equation [8] is the equation of condition for perpendicular lines.

To find the angle between two given lines, solve their equations with respect to y; the coefficients of x will be the values of a' and a, which substitute in equation [6]. Having found the tangent of α, the angle can be found as explained in Article **10**. Thus, let it be required to find the angle between the lines whose equations are,

$$3x + 2y - 12 = 0,$$

and,

$$4x + y - 6 = 0.$$

Solving with respect to y, we have,

$$y = -\frac{3}{2}x + 6,$$

and,

$$y = -4x + 6.$$

Here,

$$a' = -\frac{3}{2},$$

and,

$$a = -4;$$

hence,

$$\tan\alpha = \frac{-\frac{3}{2} + 4}{1 + 6} = \frac{5}{14}; \quad \therefore \quad \alpha = 19^\circ\, 39'.$$

In this and in the following examples the angle has only been found to the nearest minute.

EXAMPLES.

1. Find the angle between the lines whose equations are,

$$2x - 9y - 2 = 0,$$

and,

$$3x - 5y - 20 = 0.$$

Ans. $\alpha = 18^\circ\ 26'$.

2. Find the angle between the lines whose equations are,

$$5x + 4y - 52 = 0,$$

and,

$$3x + 7y - 45 = 0.$$

Ans. $\alpha = 28^\circ\ 9'$.

3. Find the angle between the lines whose equations are,

$$\frac{x}{2} + \frac{y}{3} - 12 = 0,$$

and,

$$\frac{x}{3} + \frac{y}{2} - 13 = 0.$$

Ans. $a = 22° 37'$.

By means of formulas [6], [7], and [8], a line already subjected to the condition of passing through a given point, may also be made to form a given angle with a given line.

Let it be required to find a line that shall pass through the point (3, 4), and make an angle of 45° with the line whose equation is,

$$4x - 2y + 5 = 0.$$

Solving this equation with respect to y, we have,

$$y = 2x + \frac{5}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Making $x' = 3$ and $y' = 4$, in equation [4], we have for the equation of any line through (3, 4),

$$y - 4 = a(x - 3) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Making $\tan \alpha = 1$ and $a' = 2$, in formula [6], we have,

$$1 = \frac{2 - a}{1 + 2a}; \quad \therefore \quad a = \frac{1}{3}.$$

Substituting this value of a in (2), we have,

$$y - 4 = \frac{1}{3}(x - 3),$$

which is the equation of the required line.

To make the line (2) parallel to the line (1), we have simply to make,

$$a = 2;$$

whence,

$$y - 4 = 2(x - 3).$$

To make the line (2) perpendicular to the line (1), we make,

$$a = -\frac{1}{a'} = -\frac{1}{2};$$

whence,

$$y - 4 = -\frac{1}{2}(x - 3).$$

PROBLEMS.

1°. Find the equation of a line through the point $(3, -4)$, that shall make an angle of 45° with the line

$$5x - 4y - 52 = 0.$$

Ans. $y + 4 = \frac{1}{9}(x - 3)$.

2°. Find the equation of a line passing through the point $(3, -4)$, and parallel to the line,

$$5x - 4y - 52 = 0.$$

Ans. $y + 4 = \frac{5}{4}(x - 3)$.

3°. Find the equation of a line passing through the point $(4, 1)$, and perpendicular to the line,

$$y + 4 = \frac{5}{4}(x - 3).$$

Ans. $y - 1 = -\frac{4}{5}(x - 4)$.

4°. What condition will make the line,

$$y = ax + b,$$

perpendicular to the line,

$$5x - 4y - 52 = 0\,?$$

Ans. $a = -\frac{4}{5}$.

5°. Find the shortest distance from the point (10, 2.9) to the line,

$$5y - 4x + 5 = 0.$$

The equation of a line through (10, 2.9), perpendicular to the given line, is,

$$y - 2.9 = -\frac{5}{4}(x - 10).$$

This line intersects the given line in the point (8, 5.4).

The distance between the points,

$$(10, 2.9) \quad \text{and} \quad (8, 5.4)$$

is equal to,

$$\sqrt{(2)^2 + (-2.5)^2} = \sqrt{10.25} = 3.2. \quad \textit{Ans.}$$

6°. Find the perpendicular distance from the point $(1, -2)$ to the line,

$$x + y = 3.$$

Ans. $2\sqrt{2}$.

7°. Find the angle between the lines,

$$x + 3y = 1,$$

and,

$$x - 2y = 1.$$

Ans. 45°.

8°. Find the distance from the origin to the line,

$$3x + 4y + 20 = 0.$$

Ans. 4.

9°. Find the equations of the sides of the triangle whose vertices are,

$$(-4, -1), \quad (2, 1), \quad \text{and} \quad (3, -2).$$

$$\textit{Ans.} \quad \begin{aligned} 3x + y &= 7 \\ x + 7y &= -11 \\ x - 3y &= -1. \end{aligned}$$

10°. Find the perpendicular distance from each vertex of the preceding triangle to the opposite side.

Ans. $2\sqrt{2}$, $\sqrt{10}$, and $2\sqrt{10}$.

11°. Find the vertices of the triangle whose sides are given by the equations,

$$\begin{aligned} 3x + y &= 2 \\ x + 2y &= 5 \\ 2x - 3y &= -7. \end{aligned}$$

Ans. $\left(-\frac{1}{5}, \frac{13}{5}\right)$, $\left(\frac{1}{7}, \frac{17}{7}\right)$, and $\left(-\frac{1}{11}, \frac{25}{11}\right)$.

12°. Find the equation of the line passing through the point $(2, 3)$ and the intersection of the lines,

$$\begin{aligned} 2x + 3y &= -1 \\ 3x - 4y &= 5. \end{aligned}$$

Ans. $64x - 23y = 59$.

IV. Transformation of Co-ordinates.

Definitions of Terms.

19. It is often desirable to change the reference of points from one system of co-ordinates to another; this change is effected by means of *formulas.*

The system *from* which we pass, is called the *primitive system;* that *to* which we pass, is called the *new system.* The co-ordinates of a point referred to the primitive system are called *primitive co-ordinates;* the co-ordinates of a point referred to the new system are called *new co-ordinates.*

The formulas for passing from the primitive to the new system give the values of the primitive co-ordinates of any point, in terms of the new co-ordinates of the same point, and also of certain arbitrary constants, which are called *the elements of the new system.*

Proposition 9.—*To deduce formulas for passing from a rectangular system to an oblique system.*

20. Let AX and AY be the *primitive axes;* A′X′ and A′Y′ the *new* axes; and P any point in their plane. Let AO, OP, be the *primitive* co-ordinates of P; A′O′, O′P, the *new* co-ordinates of P; AE, EA′, the co-ordinates of the new origin; and let FO′ be parallel to AY, and let A′Q and O′C be parallel to AX.

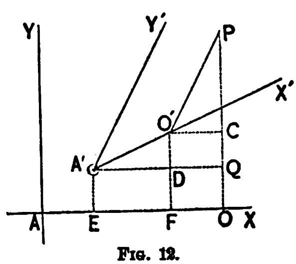

FIG. 12.

Denote AO and OP by x and y; A′O′ and O′P, by x' and y'; AE and EA′, by m and n; the inclination of A′X′, equal to QA′X′, by a; and the inclination of A′Y′, equal to CO′P, by a'.

From the figure, we have,

$$AO = AE + EF + FO,$$

and,

$$OP = OQ + QC + CP;$$

but,

$$AO = x, \quad AE = m, \quad EF = x' \cos a, \quad FO = y' \cos a';$$

and,

$$OP = y, \quad OQ = n, \quad QC = x' \sin a, \quad CP = y' \sin a'.$$

Substituting these values, we have,

$$x = m + x' \cos a + y' \cos a' \quad . \quad . \quad . \quad . \quad [9]$$

$$y = n + x' \sin a + y' \sin a' \quad . \quad . \quad . \quad . \quad [10]$$

which are the required formulas.

In equations [9] and [10], the quantities m, n, a, and a' are *elements* of the new system; by giving them suitable values, the new system may be made to fulfill any reasonable conditions. By giving proper values to m and n, the new origin may be made to coincide with any point of the plane; by giving proper values to a and a' the new axes may be made to have any inclination to the primitive axis of x.

Proposition 10.—*To deduce formulas for passing from one rectangular system to another.*

21. If we make,

$$a' - a = 90°,$$

the new axes will be perpendicular to each other. This supposition gives,

$$a' = 90° + a, \quad \therefore \quad \cos a' = -\sin a,$$

and,

$$\sin a' = \cos a.$$

Making these changes in [9] and [10], we have,

$$x = m + x' \cos a - y' \sin a \quad . \quad . \quad . \quad . \quad [11]$$

$$y = n + x' \sin a + y' \cos a \quad . \quad . \quad . \quad . \quad [12]$$

which are the required formulas.

In these equations m, n, and a are *arbitrary constants.* If we make $a = 0$ in [11] and [12], they become,

$$x = m + x' \quad . \quad . \quad . \quad . \quad . \quad . \quad [13]$$

$$y = n + y' \quad . \quad . \quad . \quad . \quad . \quad . \quad [14]$$

which are formulas for passing from a rectangular system to a parallel rectangular system.

Proposition 11.—*To deduce formulas for passing from a rectangular, to a polar system.*

22. Let AX and AY be the primitive axes; P any point in their plane; A′ the pole; and A′X′ the initial line of the new system.

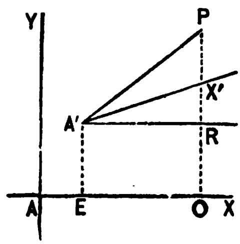

FIG. 13.

Let AO and OP be the primitive co-ordinates of P; AE and EA′, the co-ordinates of A′; X′A′P and A′P the polar co-ordinates of P; and let A′R be parallel to AX.

Denote AO and OP, by x and y; AE and EA′ by m and n; X′A′P by ϕ; A′P by r; and RA′X′ by a.

From the figure, we have,

$$AO = AE + EO,$$

and,

$$OP = OR + RP;$$

but,

$$AO = x, \quad AE = m, \quad EO = r\cos(\phi + a);$$

$$OP = y, \quad OR = n, \quad RP = r\sin(\phi + a).$$

Substituting these values, we have,

$$x = m + r\cos(\phi + a) \quad . \quad . \quad . \quad . \quad . \quad [15]$$

$$y = n + r\sin(\phi + a) \quad . \quad . \quad . \quad . \quad . \quad [16]$$

which are the required formulas.

In [15] and [16] m, n, and a are the arbitrary elements of the new system.

Proposition 12. — *To find the equation of a straight line referred to oblique axes, the origin being unchanged, and the new axis of x coinciding with the primitive one.*

23. Assume the equation,

$$y = ax + b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [3]$$

Making, m, n, and a equal to 0 in equations [9] and [10], they become,

$$x = x' + y'\cos a',$$

$$y = y'\sin a'.$$

Substituting these in [3], we have,

$$y' \sin a' = a(x' + y' \cos a') + b.$$

Solving with respect to y' and dropping the dashes from x' and y', because they are general co-ordinates, we have,

$$y = \frac{a}{\sin a' - a \cos a'} x + \frac{b}{\sin a' - a \cos a'} \quad . \quad . \quad (1)$$

which can be written under the form,

$$y = a'x + b' \quad . \quad . \quad . \quad . \quad . \quad . \quad [17]$$

Hence, the required equation is of the same form as that of a straight line referred to rectangular axes.

If we make, $x = 0$, in [17], we find $y = b'$; hence, b' is *the new intercept.*

If we substitute for a, its value, $\tan \theta$, (Art. **11**), remembering that,

$$\tan \theta = \frac{\sin \theta}{\cos \theta},$$

and then multiply both numerator and denominator by $\cos \theta$, we have,

$$a' = \frac{\sin \theta}{\sin a' \cos \theta - \cos a' \sin \theta} = \frac{\sin \theta}{\sin (a' - \theta)}.$$

Hence, a', *is the ratio of the sines of the angles that the line makes with the new co-ordinate axes.*

By a process entirely analogous to that employed in Article **16**, we can deduce the equation of a straight line through *one* point, and also the equation of a straight line through *two* points, when referred to oblique axes, as follows:

$$y - y' = a'(x - x') \quad . \quad . \quad . \quad . \quad [17a]$$

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \quad . \quad . \quad . \quad [17b]$$

These equations are the same in form as [4] and [5], but the coefficient of the second member, in each, is the *ratio of the sines of the angles that the line makes with the new axes.*

It is to be observed that all the formulas for transformation are of the first degree; hence, the degree of an equation will not be changed in passing from one system to another.

V. Of the Circle.

Definition of Terms.

24. *The locus of a point* is the path described by that point, when moving according to some fixed law. The moving point is called the *generatrix* of the line.

The locus of a line is the surface described by that line, when moving according to some fixed law. The moving line is called the *generatrix* of the surface, and any position of the generatrix is called an *element* of the surface.

The locus of an equation is a line, or surface, such that the co-ordinates of all its points satisfy the equation. Thus, *the locus of the general equation of the first degree between two variables, is a straight line.*

A *circle* is a plane curve that may be generated by a point, moving so as always to be at a fixed distance from a given point. This fixed distance is called the

radius, and the given point is called the *centre of the circle.*

A *diameter* is a straight line passing through the centre and limited by the curve. The points in which a diameter meets the curve are called *vertices* of the diameter. The horizontal diameter is called the *principal diameter,* and its left hand vertex is called the *principal vertex of the circle.*

In Plane Geometry, the term *circle* is used to denote the area bounded by the curve, which is called *the circumference;* in Analytical Geometry, the term is applied indifferently, both to the area and to the bounding curve.

Proposition 13.—*To find the equation of a circle referred to rectangular axes.*

25. Let AX and AY be the axes; C the centre of the circle; P any position of the generatrix; and let CD be parallel to AX.

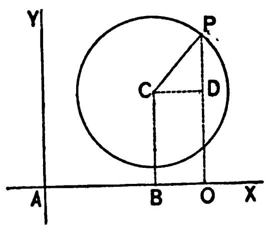

FIG. 14.

Denote the co-ordinates, AO and OP, by x and y; the co-ordinates, AB and BC, by m and n; and the radius CP by r.

From the figure, we have,

$$\overline{CD}^2 + \overline{DP}^2 = \overline{CP}^2;$$

but,

$$CD = x - m, \quad DP = y - n, \quad \text{and} \quad CP = r.$$

Substituting these values, we have,

$$(x - m)^2 + (y - n)^2 = r^2 \quad . \quad . \quad . \quad . \quad [18]$$

This equation is true for every position of P; hence, it expresses the relation between the co-ordinates of every point of the circle; *it is, therefore, the equation of the circle, which was to be found.*

The quantities, m, n, and r, are arbitrary constants; by giving them suitable values, equation [18] may be made to represent any circle that can be drawn in the plane of the co-ordinate axes. To illustrate, let it be required to find the equation of a circle that shall pass through the three points (2, 8), (5, 7), and (7, 3). Substituting these, in succession, for x and y, in [18], we find the *equations of condition* that cause the circle to pass through the given points, as follows (Art. **15**):

$$(2 - m)^2 + (8 - n)^2 = r^2$$
$$(5 - m)^2 + (7 - n)^2 = r^2$$
$$(7 - m)^2 + (3 - n)^2 = r^2$$

Combining these equations, we find,

$$m = 2, \quad n = 3, \quad \text{and} \quad r = 5.$$

Substituting these in [18], we have,

$$(x - 2)^2 + (y - 3)^2 = 25,$$

which is the required equation.

Other Forms of the Equation of a Circle.

26. If we make $m = r$ and $n = 0$, in equation [18], it reduces to,

$$(x - r)^2 + y^2 = r^2,$$

which can be placed under the form,

$$y^2 = 2rx - x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [19]$$

This is the equation of the circle when the axis of x coincides with the *principal diameter,* and when the origin is at the *principal vertex;* it is called *the equation of the circle referred to its principal vertex.*

If we make $m = 0$ and $n = 0$, in equation [18], it reduces to the form,

$$x^2 + y^2 = r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad [20]$$

This is the equation of the circle when both co-ordinate axes coincide with the diameters of the circle; it is called *the equation of the circle referred to its centre.*

EXAMPLES.

1. Find the equation of a circle through the points,

$$(-6, -1), \quad (0, 0), \quad \text{and} \quad (0, -1).$$

Ans. $(x + 3)^2 + (y + \frac{1}{2})^2 = 9\frac{1}{4}$.

2. Find the equation of a circle passing through the points,

$$(-2, 5), \quad (4, -6) \quad \text{and} \quad (-2, -6).$$

Ans. $(x - 1)^2 + (y + \frac{1}{2})^2 = 39\frac{1}{4}$.

3. Find the equation of a circle referred to its principal vertex, that shall pass through the point (2, 3).

Ans. $y^2 = \dfrac{13}{2}x - x^2$.

4. Find the equation of a circle referred to its centre, that shall pass through the point (4, 3).

Ans. $x^2 + y^2 = 25$.

Discussion of the Equations.

27. The *discussion* of an equation consists in making different hypotheses on the quantities that enter it, and then interpreting the results.

1°. To find *the points in which the circle intersects the axis of x*, we make $y = 0$ in the general equation of the circle, and find the corresponding values of x.

Making $y = 0$, in [18], and solving, we have,

$$x = m \pm \sqrt{r^2 - n^2}.$$

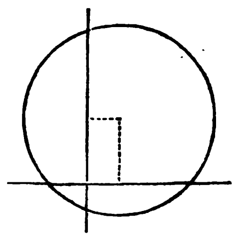

FIG. 15.

If, $n^2 < r^2$, the quantity under the radical sign is *positive*, and the two values of x are *real* and *unequal;* this shows that the curve cuts the axis of x at two points.

If, $n^2 = r^2$, the two values of x are *real* and *equal;* this shows that the curve touches the axis of x in a single point; that is, the axis of x is tangent to the curve. In all similar cases, *equality of values indicates tangency.*

If, $n^2 > r^2$, the quantity under the radical sign is *negative*, and the two values of x are *imaginary;* this shows that the curve does not intersect the axis of x.

In like manner, it may be shown that the curve cuts the axis of y in two points, if, $m^2 < r^2$; that the axis of y is tangent to the curve, if, $m^2 = r^2$; and that the curve does not cut the axis of y, if, $m^2 > r^2$.

The values of m and n that place the origin at the principal vertex, also satisfy the conditions that cause the axis of x to intersect the curve in two points, and the axis of y to be tangent to it.

The values of m and n that place the origin at the centre, make each axis intersect the curve in two points.

These results may be obtained directly, together with the values of the co-ordinates of the points, by the discussion of equations [19] and [20].

2°. *To find the limits and position of the curve with respect to the co-ordinate axes,* we discuss equation [20]. Solving this equation with respect to y, we have,

$$y = \pm \sqrt{r^2 - x^2}.$$

If, $x^2 < r^2$, that is, for any value of x, between $-r$ and $+r$, the two values of y are real; this shows that for every abscissa greater than $-r$ and less than $+r$, there are two points of the curve, and because the corresponding values of y are *numerically* equal, with contrary signs, the curve is symmetrically situated with respect to the axis of x.

If, $x^2 = r^2$, that is, if x is equal to $-r$, or to $+r$, the two values of y are equal to 0; this shows that the two lines drawn through the extremities of the principal diameter parallel to the axis of y, are tangent to the curve.

If, $x^2 > r^2$, that is, for all values of x less than $-r$ and greater than $+r$, the corresponding values of y are imaginary; this shows that there are no points of the curve lying without the two tangents just described;

these tangents, therefore, limit the curve in the direction of the axis of x.

In like manner it can be shown that the curve is symmetrical with respect to the axis of y, and that it is limited in the direction of that axis by tangents drawn through the extremities of the vertical diameter, parallel to the axis of x.

If the origin is taken at the principal vertex, the axis of x is a line of symmetry, but the axis of y is not such a line. In the general case (equation [18]), neither axis is an axis of symmetry.

3°. *To find the relation between the ordinates of any two points of the curve,* we may employ either equation [19], or [20]. Taking the latter, we place it under the form,

$$y^2 = (r + x)(r - x).$$

Let (x', y') and (x'', y'') be two points of the curve. The equations of condition that place these points on the circle are (Art. **15**),

$$y'^2 = (r + x')(r - x'),$$

and

$$y''^2 = (r + x'')(r - x'').$$

Forming a proportion from these equations, we have,

$$y'^2 : y''^2 :: (r + x')(r - x') : (r + x'')(r - x'') \quad . \quad [21]$$

But, $r + x'$ and $r - x'$, are the two segments into which the ordinate y' divides the principal diameter; and $r + x''$ and $r - x''$ are the segments into which the ordinate y'' divides the same diameter; hence, *the squares of any two ordinates of the curve, are to each*

other, as the rectangles of the segments into which they divide the principal diameter.

Proposition 14.—*To find the polar equation of the circle referred to the principal vertex.*

28. The rectangular equation of the circle referred to the principal vertex is

$$y^2 = 2rx - x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [19]$$

Making, m, n, and a equal to 0, in equations [15] and [16], and at the same time replacing r, in these equations, by ρ, to distinguish the radius-vector from the radius of the circle, we have, for making the particular transformation, the equations,

$$x = \rho \cos\phi,$$

and,

$$y = \rho \sin\phi.$$

Substituting these values in [19], we have,

$$\rho^2 \sin^2\phi = 2r\rho \cos\phi - \rho^2 \cos^2\phi.$$

Dividing through by ρ and reducing by the relation,

$$\sin^2\phi + \cos^2\phi = 1,$$

we have,

$$\rho = 2r \cos\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [22]$$

which is the required polar equation.

For every value of ϕ from 0° to 90°, and from 270° to 360°, there is one positive value for ρ, and consequently one point of the curve. For all other values of ϕ the value of ρ is negative; this shows that the radius-

vector must be produced backward to determine points of the curve corresponding to these values of ϕ.

Proposition 15.—*To find general equations of a tangent and normal to any curve.*

29. Let us assume the equation of a straight line passing through two points,

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \quad . \; . \; . \quad [5]$$

If we impose the conditions that place both points on a given curve, equation [5] will become the equation of a secant, as PQ. The slope of the secant will be that value of $\frac{y'' - y'}{x'' - x'}$ which results from a combination of the equations of condition that place the points on the curve. Denoting this slope by $\tan\theta$, we have for the equation of the secant (Art. **16**),

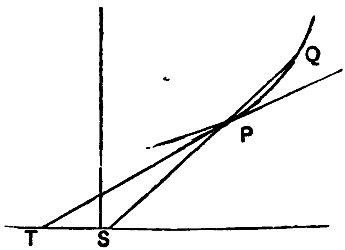

FIG. 16.

$$y - y' = \tan\theta\,(x - x').$$

If the *second* point Q, be moved along the curve toward the *first* point P, the secant PQ will revolve about P, approaching the tangent PT; when the point Q falls on P, which it will do if $x'' = x'$ and $y'' = y'$, the secant becomes a tangent to the curve at the point P, which is then called *the point of contact.* Denoting the corresponding value of the slope by $\tan\theta'$, we have, *for the general equation of a tangent to any curve,*

$$y - y' = \tan\theta'\,(x - x') \quad . \; . \; . \; . \; . \quad [23]$$

In equation [23], $\tan\theta'$, is what $\frac{y''-y'}{x''-x'}$ becomes, when $x''=x'$ and $y''=y'$.

A *normal line* to a curve is a straight line perpendicular to a tangent at the point of contact. To find the equation of a normal, assume the equation of any straight line through the point of contact,

$$y-y'=a(x-x') \quad . \quad . \quad . \quad . \quad . \quad [4]$$

The equation of condition that makes this line perpendicular to the tangent is (Art. **18**),

$$a=-\frac{1}{\tan\theta'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [8]$$

Introducing this condition into equation [4], we have, *for the general equation of a normal line to any curve,*

$$y-y'=-\frac{1}{\tan\theta'}(x-x') \quad . \quad . \quad . \quad . \quad [24]$$

The *subtangent* is the distance from the point in which the tangent intersects the axis of x, to the foot of the ordinate of the point of contact.

The *subnormal* is the distance from the foot of the ordinate of the point of contact to the point in which the normal intersects the axis of x.

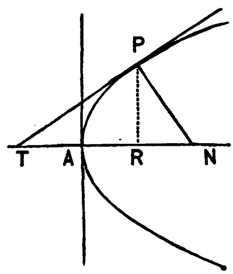

FIG. 17.

In the figure, PT is the *tangent;* PN, the *normal;* RP, the *ordinate of the point of contact;* TR, the *subtangent;* and RN, the *subnormal.*

The right-angled triangles TRP and PRN, have their

sides mutually perpendicular to each other; they are, therefore, *similar.* Hence, the angle RPN is equal to RTP, which we have denoted by θ'. The common side RP has been denoted by y'. Let us also denote the subtangent, TR, by S.T, and the subnormal, RN, by S.N. From the triangles TRP, and PRN, we deduce at once the formulas for the subtangent and subnormal as follows:

$$\text{S.T} = \frac{y'}{\tan\theta'} \quad \ldots\ldots \quad [25]$$

$$\text{S.N} = y'\tan\theta' \quad \ldots\ldots \quad [26]$$

Proposition 16.—*To find the equations of a tangent and normal to a circle.*

30. Assume the equation of the circle referred to its centre,

$$x^2 + y^2 = r^2 \quad \ldots\ldots \quad [20]$$

The equations of condition that place the points (x', y') and (x'', y'') on this curve are, (Art. **15**),

$$x'^2 + y'^2 = r^2 \quad \ldots\ldots \quad (1)$$

$$x''^2 + y''^2 = r^2 \quad \ldots\ldots \quad (2)$$

Subtracting (1) from (2), transposing and factoring, we have,

$$(y'' - y')(y'' + y') = -(x'' - x')(x'' + x') \quad \ldots \quad (3)$$

Dividing both members of (3) by $(y'' + y')(x'' - x')$, we have,

$$\frac{y'' - y'}{x'' - x'} = -\frac{x'' + x'}{y'' + y'} \quad \ldots\ldots \quad (4)$$

The second member of (4) is the slope of the secant, denoted by $\tan\theta$. If we make $x''=x'$, and $y''=y'$, we find for the slope of the tangent,

$$\tan\theta' = -\frac{x'}{y'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Substituting in [23], we have, for the equation of a tangent to a circle,

$$y-y' = -\frac{x'}{y'}(x-x') \quad . \quad . \quad . \quad . \quad (6)$$

Reducing equation (6) to its simplest form, we have,

$$yy' + xx' = x'^2 + y'^2.$$

Substituting for the second member its value taken from (1), we have, for the equation of a tangent to the circle,

$$yy' + xx' = r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Substituting in [24], we have for the equation of a normal to a circle,

$$y-y' = \frac{y'}{x'}(x-x') \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Substituting in [25] and [26], we find for the subtangent and subnormal,

$$\text{S.T} = -\frac{y'^2}{x'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\text{S.N} = -x' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The expression for the subnormal shows that the normal passes through the centre of the circle. The same conclusion may be reached by making $y=0$, in the equation of the normal, which gives $x=0$; this shows that the origin, $(0, 0)$, lies on the normal.

PROBLEMS.

1°. Find the points of intersection of the circle

$$x^2 + y^2 = 25,$$

and the straight line,

$$y + x + 1 = 0.$$

Ans. $(-4, 3)$ and $(3, -4)$.

2°. Find the points of intersection of the circle,

$$x^2 + y^2 = 25,$$

and the straight line,

$$3y + 4x + 25 = 0.$$

Ans. The straight line touches the circle at the point $(-4, -3)$.

3°. Find the equation of a tangent to the circle,

$$x^2 + y^2 = 16,$$

at a point whose abscisssa is $\sqrt{7}$.

Ans. $\sqrt{7}x \pm 3y - 16 = 0$

4°. Find the subtangent in the preceding case.

$$S.T = -\frac{9}{\sqrt{7}}$$

5°. Find the points of contact of the two lines passing through the point $(7, 1)$ and tangent to the circle

$$x^2 + y^2 = 25.$$

Ans. $(3, 4)$ and $(4, -3)$.

6°. A limited straight line moves so that its extremi-

ties are always in the co-ordinate axes; show that the locus of its middle point is a circle.

7°. Find the centre of the circle whose equation is,

$$y^2 + x^2 = 8(x + y).$$

This equation may be written,

$$(y - 4)^2 + (x - 4)^2 = 32.$$

Hence, the centre is at the point (4, 4).

8. Given the two circles,

$$(x - 5)^2 + (y - 4)^2 = 4,$$
$$(x - 2)^2 + (y - 1)^2 = 1;$$

to find a point such that the tangents drawn from it to the two circles shall be of equal length.

Solution.—Let (x', y') be the required point, and let t be the length of the tangent.

The distance from (x', y') to the centre of either circle, the corresponding tangent, and the radius to its point of contact, form a right-angled triangle, in which the first distance is the hypothenuse, and of which the tangent is the base. Taking the first circle, we have the square of the hypothenuse equal to $(x' - 5)^2 + (y' - 4)^2$ (Art. **9**), and the square of the perpendicular equal to 4. Hence,

$$t^2 = (x' - 5)^2 + (y' - 4)^2 - 4 \quad . \quad . \quad . \quad (1)$$

In like manner, for the second circle, we have,

$$t^2 = (x' - 2)^2 + (y' - 1)^2 - 1 \quad . \quad . \quad . \quad (2)$$

Combining (1) and (2), and reducing, we have,

$$y' = -x' + \frac{11}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But equation (3) is the equation of condition that the required point shall lie on the straight line whose equation is,

$$y = -x + \frac{11}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and furthermore the point may be anywhere on this line provided it is outside of both circles.

The line whose equation is (4), is called the *radical axis* of the given circles.

The equation of the line through the centres of the given circles is [5],

$$y - 1 = \frac{4-1}{5-2}(x-2),$$

or,

$$y = x - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Comparing (4) and (5), we see that the corresponding lines are perpendicular to each other, (Art. **18**), that is, the radical axis of the circles is perpendicular to the line joining the centres.

Any two circles have a radical axis, that is, a line such that the tangents drawn from any point of it to the two circles are equal.

9°. Let it be proved that the radical axis of two intersecting circles passes through the points of intersection.

10°. Find a point from which the tangents drawn to three circles shall be equal.

Solution.—The point lies on the radical axis of the first and second circles, also on the radical axis of the first and third circles. Hence, it is at their intersection. Let this point be found for the three circles,

$$(x - 5)^2 + (y - 6)^2 = 4,$$
$$(x - 3)^2 + (y - 1)^2 = 1,$$
$$(x + 1)^2 + (y + 2)^2 = 9.$$

Ans. $\left(-\frac{79}{28}, \frac{166}{28}\right)$.

VI. Of the Parabola.

Definitions of Terms.

31. **A Parabola** is a plane curve that may be generated by a point, moving so that it shall always be equally distant from a fixed line and a given point.

The *fixed line* is called the *directrix;* the *given point* is called the *focus*, and the straight line through the focus, perpendicular to the directrix, is called the *principal axis* of the curve.

Construction of the Curve.

32. The curve may be constructed by a continuous movement, or by points:

1°. *By continuous movement.* Let BL be the directrix, and F the focus. Take a triangular ruler, DCL, right-angled at C, and place one side, CL, on the directrix; take a string equal in length to CD, and fasten one end at F, and the other end at D; then press a

pencil against the string, keeping its point, P, against the ruler, and move the ruler along the directrix; the pencil will trace out a portion of the curve; for, in every position, we shall have,

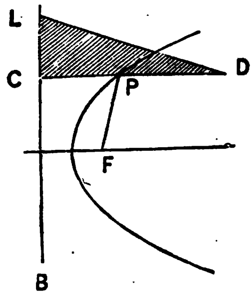

FIG. 18.

$$FP = CP.$$

2°. *By points.* Let BL be the directrix; F the focus; and DE the *principal axis* of a parabola. At any point of the axis, as E, draw EP perpendicular to it; with F as a centre and DE as a radius, describe an arc cutting the perpendicular at P and P; these will be points of the curve; for, we shall always have,

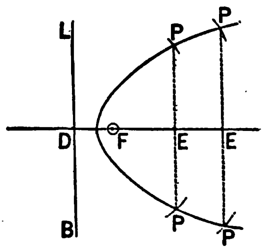

FIG. 19.

$$FP = LP:$$

having found a sufficient number of points, *draw a line through them and it will be the required curve.*

Proposition 17.—*To find the equation of a parabola.*

33. Let DC be the directrix; F the focus; and BX the principal axis of a parabola. The middle point, A, of BF, will obviously be a point of the curve. Let this point, which is called the *principal vertex* of the curve,

be taken as the origin of co-ordinates. Let AX and AY be the co-ordinate axes; P any point of the curve; AR and RP its co-ordinates.

Denote AR and RP by x and y; and the distance AF, equal to AB, by $\frac{p}{2}$.

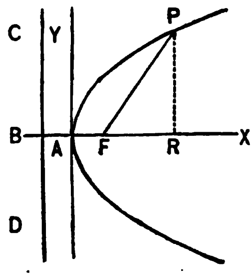

FIG. 20.

From the right-angled triangle FRP, we have,

$$\overline{RP}^2 = \overline{FP}^2 - \overline{FR}^2;$$

but,

$$RP = y, \quad FP = CP = x + \frac{p}{2},$$

and,

$$FR = AR - AF = x - \frac{p}{2},$$

Substituting in the preceding equations, we have,

$$y^2 = \left(x + \frac{p}{2}\right)^2 - \left(x - \frac{p}{2}\right)^2;$$

whence, by reduction,

$$y^2 = 2px \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [27]$$

This equation is true for every position of P; hence, *it is the equation of a parabola, which was to be found.*

Equation [27] is called *the equation of the parabola referred to its principal vertex.*

Discussion of the Equation.

34. 1°. *To find the points in which the curve inter-*

sects the axes. If we make $y = 0$ in [27], we find $x = 0$; this shows that the curve cuts the axis of x in but one point, and that point is the *principal vertex.* If we make $x = 0$ we find $y = \pm 0$; this shows that the axis of y is tangent to the curve at the principal vertex. The line AY is called the *vertical* tangent.

2°. *To find the position, limits, and extent of the curve, with respect to the co-ordinate axes.* Solving equation [27] with respect to y, we have,

$$y = \pm \sqrt{2px} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The quantity $2p$ is arbitrary and may have any value from $-\infty$ to $+\infty$. If we suppose $2p$ to be positive the values of y will be *real* for all *positive* values of x, and *imaginary* for all *negative* values of x; this shows that the curve extends to an infinite distance in the direction of positive abscissas, and that it has no point whose abscissa is negative. The curve is therefore limited in the direction of negative abscissas by the axis of y. If we suppose $2p$ to be negative, it may be shown in like manner that the curve extends to an infinite distance in the direction of negative abscissas, and that it is limited in the direction of positive abscissas by the axis of y. In what follows, we shall suppose $2p$ to be positive, unless the contrary is stated.

We see from equation (1), that for every value of x, corresponding to points of the curve, there are two values of y, numerically equal, but having contrary signs; this shows that the curve is symmetrical with respect to the axis of x.

Solving equation [27] with respect to x, we have,

$$x = \frac{y^2}{2p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For every value of y, whether positive or negative, there will be a single, and consequently a real value of x; this shows that the curve extends to an infinite distance in the direction of both positive and negative ordinates.

3°. *To find the breadth of the curve through the focus.* This breadth is called the *parameter*, and is equal to the double ordinate through the focus. Making $x = \frac{p}{2}$, in equation (1), we have $y = p$; hence, the double ordinate is equal to $2p$. But, from equation [27], we have the proportion,

$$x : y :: y : 2p \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

that is, *the parameter is a third proportional to any abscissa and the corresponding ordinate.*

If the parameter is known, the curve can be constructed by points, as follows:

Lay off AB on the axis of x, to the left, equal to $2p$; assume any abscissa AP, and on BP, as a diameter, describe a semicircle cutting the axis of y in Q; draw PM and QM parallel to the co-ordinate axes, till they intersect at M. Then will M be a point of the curve; for, from a property of the circle, we have,

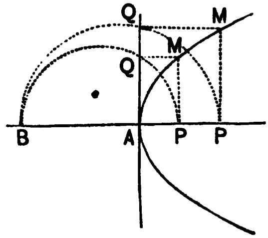

Fig. 21.

$$\overline{QA}^2 = BA \times AP;$$

but, $QA = PM$ and $BA = 2p$.

Substituting these values in the preceding equation, we have,

$$\overline{PM}^2 = 2p \times AP;$$

that is, the co-ordinates of M satisfy the equation of the parabola; hence, the point M is a point of the curve. In like manner, any number of points of the curve may be found.

4°. *To find the relation between the ordinates of any two points of the curve.* Let (x', y') and (x'', y'') be any two points of the curve. The equations of condition that place these points on the parabola are, (Art. **15**),

$$y'^2 = 2px',$$
$$y''^2 = 2px'';$$

forming a proportion from these equations, we have,

$$y'^2 : y''^2 :: x' : x'' \quad . \quad . \quad . \quad . \quad . \quad (4)$$

that is, *the squares of any two ordinates are to each other as the corresponding abscissas.*

5°. *To determine the position of any point with respect to the parabola.* Let (x', y') be the point, and let a line be drawn through it parallel to the principal axis.

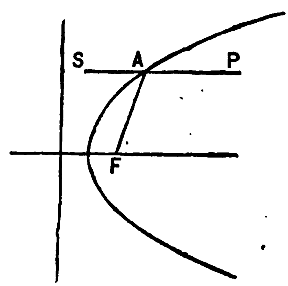

Fig. 22.

The point may be on the curve, as at A; it may be without the curve, as at S; or, it may be within the curve, as at P.

If the point is on the curve,

as at A, its co-ordinates satisfy the equation of the curve, and we have,

$$y'^2 - 2px' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If the point is without the curve, as at S, its ordinate is equal to the ordinate of A, but its abscissa is less than the abscissa of A, because S is nearer to $-\infty$ than A, and consequently, the value of $2px'$ for S is less than it is for A; that is,

$$y'^2 - 2px' > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If the point is within the curve, as at P, its ordinate is equal to the ordinate of A, but the value of $2px'$ for P is greater than it is for A; that is,

$$y'^2 - 2px' < 0 \quad . \quad . \quad . \quad . \quad . \quad , \quad . \quad (7)$$

The expressions (5), (6), and (7), enable us to determine whether a given point lies *upon*, *without*, or *within* a given parabola.

Proposition 18.—*To find the polar equation of a parabola when the pole is at the focus, and the initial line coincides with the principal axis.*

35. The rectangular equation of the parabola, referred to its principal vertex, is,

$$y^2 = 2px \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [27]$$

Making $m = \frac{1}{2}p$, $n = 0$, and $a = 0$, in [15] and [16], we have, for this particular transformation, the equations,

$$x = \frac{p}{2} + r\cos\phi \quad \text{and} \quad y = r\sin\phi.$$

Substituting in [27], we have,

$$r^2 \sin^2 \phi = p^2 + 2pr \cos \phi.$$

Substituting for $\sin^2 \phi$ its equal, $1 - \cos^2 \phi$, and transposing the term, $-r^2 \cos^2 \phi$, we have,

$$r^2 = p^2 + 2pr \cos \phi + r^2 \cos^2 \phi.$$

Extracting the square root of both members, and neglecting the negative value of r, we have,

$$r = p + r \cos \phi.$$

Solving with respect to r,

$$r = \frac{p}{1 - \cos \phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad [28]$$

which is the polar equation required.

This equation might have been deduced directly from figure 20. For, by the definition of the curve, we have,

$$\text{FP} = \text{BF} + \text{FR};$$

but,

$$\text{FP} = r, \ \text{BF} = p, \text{ and } \text{FR} = r \cos \text{XFP} = r \cos \phi.$$

Substituting these values, we have,

$$r = p + r \cos \phi;$$

whence, by solving with respect to r,

$$r = \frac{p}{1 - \cos \phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad [28]$$

Discussion of the Polar Equation.

36. If we make $\phi = 0$, in [28], we find $r = \infty$; this shows that the principal axis does not intersect the curve to the right of the focus.

As the value of ϕ increases from 0° to 180°, the corresponding value of r *diminishes.* If $\phi = 90°$, $r = p$; and if $\phi = 180°$, $r = \frac{1}{2}p$.

As the value of ϕ increases from 180° to 360°, the corresponding value of r *increases;* if $\phi = 270°$, $r = p$; and if $\phi = 360°$, r again becomes ∞.

The two values of r, corresponding to 90° and 270°, taken together, constitute the *parameter* of the curve, which is again shown equal to $2p$.

Proposition 19.—*To find the equations of a tangent and normal to a parabola.*

37. Assume the equation of the parabola, referred to its principal vertex,

$$y^2 = 2px \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [27]$$

The equations of condition that place the points (x', y') and (x'', y'') on the curve are, (Art. **15**),

$$y'^2 = 2px' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$y''^2 = 2px'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Subtracting (1) from (2), and factoring, we have,

$$(y'' - y')(y'' + y') = 2p(x'' - x') \quad . \quad . \quad (3)$$

Dividing both members of (3) by $(y'' + y')(x'' - x')$, we have,

$$\frac{y'' - y'}{x'' - x'} = \frac{2p}{y'' + y'} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The second member of (4), is the slope of the secant (Art. **29**). If we make $x''=x'$ and $y''=y'$, we find for the slope of the tangent,

$$\tan\theta'=\frac{p}{y'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Substituting in [23], we have, for the equation of the tangent,

$$y-y'=\frac{p}{y'}(x-x') \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Reducing (6) and substituting for y'^2 its value $2px'$ (Eq. 1), we have, for the equation of a tangent to the parabola,

$$yy'=p(x+x') \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Substituting in [24], we have for the equation of the normal,

$$y-y'=-\frac{y'}{p}(x-x') \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Substituting in [25] and [26], we find for the subtangent and subnormal,

$$\text{S.T}=\frac{y'^2}{p}=\frac{2px'}{p}=2x' \quad . \quad . \quad . \quad . \quad (9)$$

$$\text{S.N}=p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

From equation (9) we see that the subtangent is equal to twice the abscissa of the point of contact, that is, *the subtangent is bisected at the principal vertex.*

From equation (10) we see that, *the subnormal is constant and equal to half the parameter of the curve.*

These results suggest two methods of drawing a tangent to a parabola at a given point.

First.—Let P be the given point, and RP its ordinate; lay off to the left of the origin a distance equal to AR, the abscissa of P; from the point T, thus determined, draw TP and it will be the required tangent.

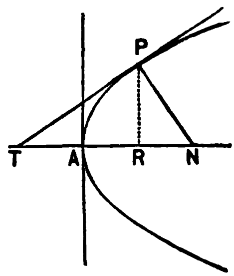

FIG. 23.

Secondly.—Lay off from the foot of the ordinate of the point of contact, toward the right, a distance, RN, equal to half the parameter, and draw NP; through P draw PT perpendicular to NP; PN will be a normal and PT will be the required tangent.

Proposition 20. — *To prove that a tangent to a parabola makes equal angles with the principal axis, and the focal line to the point of contact.*

38. A **Focal Line** is a line drawn from the focus to a point of the curve.

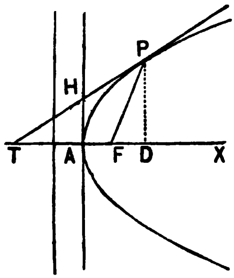

FIG. 24.

Let PT be tangent to the curve at P; FP, the focal line to the point of contact; and DP the ordinate of the point of contact.

From the figure, we have,

$$FT = FA + AT.$$

From Article **33**, we have,

$$FP = FA + AD.$$

But, AD is equal to AT, by the last proposition; hence, FT is equal to FP, and *consequently*, the angle FTP is equal to FPT, *which was to be proved.*

If a line be drawn from F, perpendicular to PT, it will intersect PT at its middle point H, because the triangle TFP is isosceles. The vertical tangent intersects PT at its middle point H, because it bisects DT and is parallel to DP. Hence, *if a line be drawn from the focus perpendicular to any tangent, the point of intersection will be on the vertical tangent.*

The principle demonstrated in this proposition, is the basis of three important constructions.

1°. *To draw a tangent to a parabola at a given point.* Let FP be the focal line to the point of contact; lay off FT to the left, equal to FP, and draw TP; then, will TP be the required tangent. For, the triangle PFT is isosceles by construction, and consequently the line TP makes equal angles with the axis and the focal line to the point of contact.

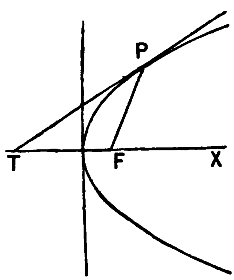

FIG. 25.

2°. *To draw a tangent to a parabola parallel to a given line.* Let BC be the given line, and F the focus. Draw the line FP, making the inclination XFP equal to

twice the inclination of BC; through the point P, in which it meets the curve, draw a line, PT, parallel to BC; then will PT be the required tangent. For, the exterior angle, XFP, is equal to the sum of the angles FTP and FPT; but, FTP is equal to the inclination of BC, that is, to half of XFP, and consequently FPT is also equal to half of XFP; hence, PT makes equal angles with the axis and the focal line to the point of contact.

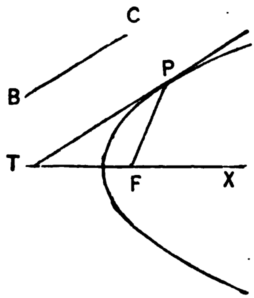

FIG. 26.

3°. *To draw a tangent to a parabola through a given point without the curve.* Let G be the given point; CC′ the directrix; and F the focus. With G as a centre and GF as a radius, describe a circle cutting the directrix in C and C′; through these points draw CP and C′P′ parallel to BX; then will GP and GP′ be tangent to the curve at P and P′. For, CP = FP, from the definition of the curve; and GF = GC, because they are radii of the same circle; hence, GP is perpendicular to CF, and consequently the angle FPT is equal to CPT; but, the angle CPT is equal to PTF because they are alternate; hence, GP makes equal angles

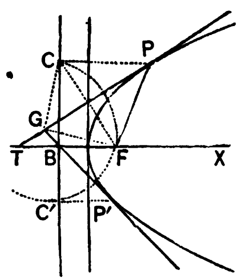

FIG. 26 A.

with FP and FT, and is therefore tangent to the curve at P.

In like manner it may be shown that GP′ is tangent to the curve at P′. If G lies without the curve, the circle GP cuts the directrix in two points, and there are two corresponding tangents. If G is on the curve, the circle is tangent to the directrix, and there is but one corresponding tangent. If G falls within the curve, the circle has no point in common with the directrix, and consequently there is no tangent possible.

Proposition 21.—*To find the ordinates of the point of contact of a tangent drawn to a parabola from a given point.*

39. Let (x'', y'') be the given point, and assume the equation of a tangent to the parabola,

$$yy' = p(x + x') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The equation of condition that puts the given point on the tangent is (Art. **15**),

$$y''y' = p(x'' + x') \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The equation of condition that puts the point of contact on the curve is (Art. **15**),

$$y'^2 = 2px' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Finding the value of px' from (3), and substituting in (2), we have,

$$y''y' = px'' + \frac{y'^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Clearing of fractions, and transposing, we have,

$$y'^2 - 2y''y' = -2px'' \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Solving with respect to y', we have,

$$y' = y'' \pm \sqrt{y''^2 - 2px''} \quad . \quad . \quad . \quad . \quad (6)$$

The point (x'', y'') is *without, upon,* or *within* the curve, according as the quantity under the radical sign is *greater than, equal to,* or *less than,* 0 (Conditions 5, 6, and 7, Art. **34**); in the first case, there are *two* tangents; in the second case there is *one* tangent; and in the third case there is *no* tangent. These results agree with those already deduced.

The ordinate of the middle point of the chord joining the two points of contact in the first case, is equal to the half sum of the ordinates of the two points; that is, it is equal to y''. Hence, a line through the given point, parallel to the principal axis, bisects the chord joining the two points of contact. This chord is called the *chord of contact.*

If we denote the two values of y', in equation (6), by y''' and y'''', their product will be equal to the second member of equation (5), with its sign changed; that is,

$$y'''y'''' = +2px'' \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If we suppose the point (x'', y'') to be anywhere on the directrix, we shall have,

$$x'' = -\frac{p}{2};$$

which being substituted in (7), gives,

$$y'''y'''' = -p^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The slopes of the two tangents through (x'', y'') are given by the equations,

$$\tan a = \frac{p}{y'''} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\tan a' = \frac{p}{y''''} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Multiplying these together, and substituting for $y'''y''''$ its value from (8), we have,

$$\tan a \tan a' = -1, \quad \text{or,} \quad \tan a = -\frac{1}{\tan a'} \quad . \ . \ (11)$$

But (11) is the equation of condition that makes the two tangents perpendicular to each other. Hence, if two tangents be drawn to a parabola from any point of the directrix, they are perpendicular to each other.

Proposition 22. — *To find the equation of the parabola referred to oblique axes, the new axis of x being parallel to the primitive one.*

40. Assume the equation of the parabola,

$$y^2 = 2px \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [27]$$

Making $a = 0$ in [9] and 10], we have the formulas for this transformation,

$$x = m + x' + y' \cos a' \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$y = n + y' \sin a' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting in [27], and arranging, we have,

$$y'^2 \sin^2 a' + (2n \sin a' - 2p \cos a')\, y' + (n^2 - 2pm) = 2px' \quad . \ . \ (3)$$

which is the required equation. In it, m, n, and a' are arbitrary elements of the new system.

If we make,

$$2n \sin a' - 2p \cos a' = 0 \quad . \quad . \quad . \quad . \quad (4)$$

and,

$$n^2 - 2pm = 0 \quad . \quad , \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

equation (3) will become,

$$y'^2 \sin^2 a' = 2px' \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Putting $\frac{2p}{\sin^2 a'} = 2p'$, and dropping the dashes on x and y, because they are general variables, we have,

$$y^2 = 2p'x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [29]$$

which is of the same general form as equation [27].

Equation (5) is the equation of condition that puts the point (m, n) on the curve; hence, the new origin is on the curve. But, equation (5) is satisfied if the point (m, n) is at any point of the curve; hence, the new origin, may be anywhere on the curve.

Equation (4) may be reduced to the form,

$$\tan a' = \frac{p}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This value of $\tan a'$ is equal to the slope of a tangent to the parabola, at the point (m, n) (Art. **37**); hence, the new axis of y is tangent to the curve at the new origin.

Discussion of the Equation.

41. It may be shown, as in Article **34**, that the curve cuts the new axis of x in but one point, and that the axis of y is tangent to the curve at the new origin. It may also be shown that the curve extends to an infi-

nite distance in the direction of positive abscissas, and is limited in the direction of negative abscissas by the axis of y; also that the curve extends to an infinite distance in the direction of both positive and negative ordinates.

If we solve equation [29] with respect to y, we have,

$$y = \pm \sqrt{2p'x} \quad . \; . \; (1)$$

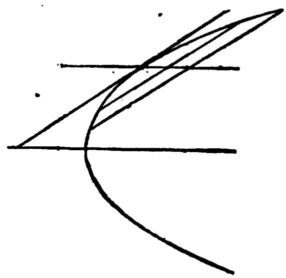

FIG. 27.

From this equation, we see that for each positive value of x there are two real values of y, numerically equal, but having contrary signs. These values, taken together, make up a chord parallel to the axis of y, and which is bisected by the axis of x. Hence, the axis of x bisects all chords of the curve parallel to the axis of y.

A *diameter* of a curve is a straight line that bisects a *system* of parallel chords. The point in which a diameter meets the curve is called the *vertex* of the diameter.

If a diameter meet a curve in two points, it has two *vertices.*

The new axis of x is therefore a diameter of the parabola; and because any line parallel to the principal axis of the curve may be taken as the axis of x, it follows that the parabola has an infinite number of diameters, all parallel to the axis, and each bisecting a system of chords parallel to the tangent at its vertex. These diameters are *lines of symmetry*; but the axis is the

only one that is a line of *right symmetry*, all the rest being lines of *oblique symmetry*.

It may be shown, as before, that the squares of any two ordinates are to each other as their corresponding abscissas.

The quantity of $2p'$, or its equal, $\frac{2p}{\sin^2 a'}$, is called the *parameter* of the diameter that is taken as the axis of x.

Proposition 23.—*To find the equation of a tangent to a parabola referred to a diameter and tangent.*

42. We have seen, (Art. **23**), that the equation of a straight line referred to oblique axes is of the same form as when referred to rectangular axes. We have also seen, (Art. **40**), that the equation of the parabola referred to *any vertex*, is of the same form as when referred to the *principal vertex*. Hence, by a course of reasoning identical with that employed in deducing the equation of a tangent referred to rectangular axes, we find the required equation,

$$yy' = p'(x + x') \quad . \quad . \quad . \quad . \quad . \quad [30]$$

If we make $y = 0$, in [30] we find, $x = -x'$: this shows that the tangent intersects the axis of x on the left of the new origin, and at a distance from it equal to the abscissa of the point of contact; hence, in this case also, *the subtangent is bisected at the vertex.*

Of Poles and Polars.

43. Let (m, n) be a point, referred to the same axes as the parabola, (Art. **40**), and assume the equation of a tangent, also referred to the same axes,

$$yy' = p'(x + x') \quad . \quad . \quad . \quad . \quad . \quad . \quad [30]$$

The equation of condition that causes this tangent to pass through the point (m, n) is, (Art. **15**),

$$ny' = p'(m + x') \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But it has been shown, (Art. **39**), that two tangents can always be drawn to a parabola from a point without the curve; hence there are two sets of values of x' and y', that will satisfy equation (1), and at the same time satisfy the equation of the curve. The quantities x' and y' in equation (1) are therefore the co-ordinates of both points of contact.

Equation (1) is obviously the equation of condition that places the two points of contact on the line whose equation is,

$$ny = p'(m + x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Because equation (2) is satisfied by the co-ordinates of both points of contact, it must be the equation of the straight line passing through these points, that is, *it is the equation of the chord of contact.*

If we make $y = 0$ in equation (2), we find,

$$x = -m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This is the abscissa of the point in which the chord of contact cuts the axis of x, and this abscissa is con-

stant, so long as m is unchanged, that is, so long as the point (m, n) remains at a fixed distance from the axis of y. If, therefore, we suppose m to remain fixed in value, whilst n takes in succession every possible value from $-\infty$ to $+\infty$, the point (m, n) will generate a line parallel to the axis of y, such that if from any point of it two tangents be drawn to the parabola, the corresponding chord of contact will pass through the point on the axis of x, whose abscissa is given by equation (3).

This line, from which the tangents are drawn, is called the *polar* of the point; and the point in which the chords of contact intersect, is called the *pole* of the line.

The terms *pole* and *polar* are correlative; that is, neither has any significance, except with respect to the other.

The point in which the polar intersects the diameter through the pole is called the *polar point*. From equation (3), we see that the pole and polar point are on opposite sides of the origin, and equally distant from it. We also see that x and m may change places in equation (3); hence, if the primitive *polar point* be taken as a *pole*, the *primitive pole* will be the new *polar point*.

It is obvious that every straight line in the plane of the curve has a corresponding pole, and that every point in the plane has a corresponding *polar*.

From the principles explained, we deduce the following constructions:

To find the pole of a given polar: Draw a tangent to the curve, parallel to the polar; draw also a diameter through the point of contact, and produce it to intersect

the polar; having found the *polar point*, lay off from the vertex of the diameter, in the opposite direction, a distance equal to that of the polar point; the point thus found is the *pole*.

To find the polar of a given pole: Draw a diameter through the pole, and produce it until the prolongation is equal to the distance of the pole from the vertex; the point thus found is the *polar point;* draw a tangent to the curve at the vertex, and through the polar point draw a line parallel to the tangent; this is the *polar*.

If the polar cuts the curve, the pole is without the parabola, and the chords of contact of the different pairs of tangents must be prolonged to pass through it; in this case, only those points of the polar that lie without the curve, satisfy the definition of a polar.

The focus is the pole of the directrix, and conversely, the directrix is the polar of the focus.

PROBLEMS.

1°. Find the intersections of the parabola $y^2 = 8x$ and the straight line $3y - 2x - 8 = 0$.

Ans. $(2, 4)$ and $(8, 8)$.

2°. Find the equation of a straight line passing through the focus of the parabola $y^2 = 4x$, and making an angle of 45° with the axis.

Ans. $y = x - 1$.

3°. Find the points in which the *focal chord*, $y = x - 1$, intersects the parabola $y^2 = 4x$.

Ans. $(3 \pm 2\sqrt{2}, 2 \pm 2\sqrt{2})$.

4°. Find the equations of a tangent and normal to the parabola, $y^2 = 4x$, at the extremity of the positive ordinate through the focus.

Ans. $y = x + 1$ and $y = -x + 3$.

5°. Find the equation of a straight line passing through the vertex of the parabola, $y^2 = 4x$, and the extremity of the *focal* ordinate.

Ans. $y = 2x$.

6°. Find the equation of a circle passing through the vertex of the parabola, $y^2 = 8x$, and the extremities of the double ordinate through the focus.

Ans. $y^2 = 10x - x^2$.

7°. Find the point in which the normal, $y = -x + 3$, to the parabola, $y^2 = 4x$, again cuts the curve, and also the length of the intercepted chord.

Ans. $(9, -6)$; chord $= 8\sqrt{2}$.

8°. Find the point of the parabola, $y^2 = 6x$, at which a tangent makes an angle of 30° with the axis.

Ans. $(4\frac{1}{2}, 3\sqrt{3})$.

9°. Show that the tangents to any parabola at the extremities of a focal chord, are perpendicular to each other.

Solution.—Combining the equation of the parabola,

$$y^2 = 2px,$$

with the equation of a focal chord,

$$y = a\left(x - \frac{p}{2}\right),$$

we have the equation,

$$y^2 - \frac{2p}{a}y = p^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Denoting the two values of y by y' and y'', we know from algebra that,

$$y'y'' = -p^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The slope of the tangent at the point whose ordinate is y' (Art. **37**), is $\frac{p}{y'}$, and the slope of the tangent at the point whose ordinate is y'', is $\frac{p}{y''}$; hence, the product of their slopes is $\frac{p^2}{y'y''}$; or, replacing $y'y''$ by its value taken from equation (2), we have,

$$\tan\theta' \tan\theta'' = -1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence, (Art. **18**), the tangents are perpendicular to each other.

10°. Find a formula for the distance from a point (x', y') to any point of the parabola $y^2 = 2px$.

$$Ans. \quad d = \sqrt{(y' - \sqrt{2px})^2 + (x' - x)^2}.$$

11°. Find the conditions that will make this the value of d rational.

Solution.—The quantity under the radical sign, when developed, takes the form,

$$y'^2 - 2y'\sqrt{2px} + 2px + x'^2 - 2x'x + x^2 \quad . \quad . \quad (1)$$

In order that this may be a perfect square for all values of x, the second term must reduce to 0; that is, we must have,

$$y' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This supposition enables us to place (1) under the form,

$$x'^2 + 2(p - x')x + x^2 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In order that (2) may be a perfect square, the middle term must be equal to twice the product of the square roots of the extremes; that is,

$$2(p - x')x = 2x'x;$$

whence,

$$x' = \frac{p}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From (2) and (4), we see that *the required conditions place the point* (x', y') *at the focus.*

12°. Find the length of a focal chord in terms of its inclination to the axis.

Solution.—Denote the inclination by a; make $\phi = a$, also to $180° + a$, in the polar equation of the curve, and take the sum of the resulting values of r; this gives,

$$r' + r'' = \frac{p}{1 - \cos a} + \frac{p}{1 + \cos a} = \frac{2p}{\sin^2 a}. \quad \textit{Ans.}$$

The focal chord is the parameter of the diameter that bisects it, (Art. **41**).

13°. Show that the circle described on a focal chord as a diameter, is tangent to the directrix.

Solution.—From the definition of the curve, the sum of the distances of the extremities of the focal chord from the directrix, is equal to the length of the focal chord; hence, the middle of that chord is at a distance from the directrix equal to one-half the focal chord. If,

therefore, a circle be described, with this distance as a radius, it will be tangent to the directrix, and will pass through the extremities of the focal chord.

14°. Show that the parameter of any diameter of the curve is equal to four times the distance of its vertex from the focus.

Solution.—Let a focal chord be drawn parallel to the tangent at the vertex of the given diameter, and let a circle be described on this chord, as a diameter. The radius of this circle, drawn to the point of contact of the circle and directrix, is bisected by the curve; but, this point of the curve is equally distant from the directrix and the focus. Hence, the focal chord is equal to four times the distance from the vertex of the given diameter to the focus. But, the focal chord is equal to the parameter of the diameter that bisects it, (Problem 12°). Consequently, the parameter of any diameter is equal to four times the distance from its vertex to the focus.

If we draw two tangents to the parabola at the extremities of the focal chord, they will intersect on the directrix, at the point of contact of the circle described on the focal chord.

VII. Of the Ellipse.

Definitions of Terms.

44. An **Ellipse**, is a plane curve, that may be generated by a point, moving so that the *sum* of its distances from two *fixed points* is equal to a given distance.

The fixed points are called *foci*. *The transverse axis*, is a straight line through the foci, and limited by the curve. The *centre*, is a point on the transverse axis midway between the foci. The *conjugate axis*, is a straight line through the centre, perpendicular to the transverse axis, and limited by the curve. A *diameter*, is any straight line through the centre limited by the curve. The points in which a diameter meets the curve are called *vertices*. The left-hand vertex of the transverse axis is called the *principal vertex* of the ellipse. The *excentricity* of an ellipse is the distance from its centre to either focus, divided by the semi-transverse axis.

Construction of the Curve.

45. The curve may be constructed by a continuous movement, or by points.

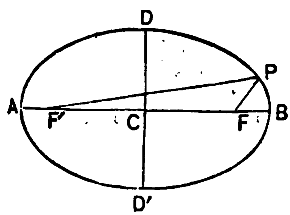

FIG. 28.

1°. *By continuous movement.* Let F, F′, be the foci and C the centre. Take a string longer than FF′, and fasten one end at F and the other at F′; then press a pencil, P, against the string and move it around F and F′, keeping the string tense; then will P generate an ellipse. For in every position of the pencil the sum of the distances, PF and PF′, is equal to the length of the string, that is, it is constant.

The line AB is the transverse axis, and DD′ perpen-

dicular to it through C, is the conjugate axis. Because B is a point of the curve, we have,

$$BF + BF',$$

or,

$$FF' + 2FB,$$

equal to the length of the string; for a like reason we have,

$$FF' + 2AF',$$

equal to the length of the string; hence, FB is equal to F′A, CB to CA, and AB to length of the string.

Because DD′ is perpendicular to FF′, at its middle point, either of the points, D, D′, is equidistant from F and F′; that is, either vertex of the conjugate axis is at a distance from either focus, equal to the semi-transverse axis.

2°. *By points.* Suppose the axes to be given. From D, as a centre, with CB as a radius, describe an arc cutting AB in F and F′; these points are the *foci.* Next, take a radius greater than the distance from A to the nearer focus, and less than the distance to the remote focus, and with one focus as a centre, describe an arc; then with the remainder of the transverse axis as a radius, and with the other focus as a centre, describe another arc cutting the first, in two points; these will be points of the curve. For, the sum of the focal distances of each point, is equal to the transverse axis.

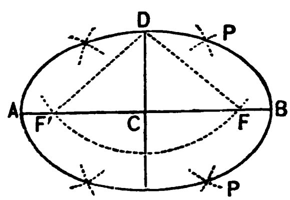

FIG. 29.

Having found a sufficient number of points, draw a curve through them, and it will be the required ellipse.

Proposition 24. — *To find the equation of an ellipse referred to its centre.*

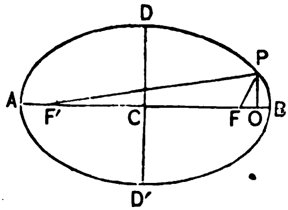

Fig. 30.

46. Let C be the centre, AB and DD′ the axes; and P any point of the curve. Denote the focal lines, FP, by r, and F′P by r', the semi-transverse axis by a, the semi-conjugate axis by b, the distance, CF, or CF′, by c, and the co-ordinates of P by x and y. Then will FO be equal to $x - c$, and, F′O to $x + c$.

From the right-angled triangles FOP and F′OP, we have,

$$r^2 = y^2 + (x - c)^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$r'^2 = y^2 + (c + x)^2 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From the definition of the curve, we have,

$$r' + r = 2a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Adding (1) and (2), and subtracting (1) from (2), we have,

$$r'^2 + r^2 = 2\,(x^2 + y^2 + c^2) \quad . \quad . \quad . \quad . \quad (4)$$

$$r'^2 - r^2 = 4cx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Factoring (5), and then dividing it by (3), member by member, we have,

$$r' - r = \frac{2cx}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Combining (3) and (6), we find for the *focal lines*,

$$r' = a + \frac{cx}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$r = a - \frac{cx}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Squaring (7) and (8), and substituting in (4), we have,

$$2a^2 + \frac{2c^2}{a^2}x^2 = 2(x^2 + y^2 + c^2) \quad . \quad . \quad . \quad (9)$$

Reducing, we have,

$$y^2 + \frac{a^2 - c^2}{a^2}x^2 = a^2 - c^2 \quad . \quad . \quad . \quad . \quad (10)$$

But, because the distance FD is equal to a, we have,

$$a^2 - c^2 = b^2;$$

substituting this in (10), and multiplying by a^2, we have,

$$a^2y^2 + b^2x^2 = a^2b^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad [31]$$

This equation is true for every position of the generatrix; hence, *it is the required equation.*

If we divide both members of [31], by a^2b^2, it may be put under the form,

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [32]$$

If we denote the excentricity by e, we have from the definition,

$$e = \frac{c}{a}, \quad \text{or} \quad c = ae; \quad \therefore \quad b^2 = a^2(1 - e^2).$$

Substituting in (7) and (8), we have,

$$r' = a + ex \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$r = a - ex \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Discussion of the Equation.

47.—1°. *To find the points in which the curve cuts the axes.*—If we make $y = 0$, in [32], we find $x = \pm a$; if we make $x = 0$, we find $y = \pm b$; hence, the curve cuts the axis of x in the points $(-a, 0)$, $(+a, 0)$, and the axis of y in the points $(0, -b)$, $(0, +b)$.

2°. *To find the position and limits of the curve.*—Solving [31] with respect to y and x, separately, we have,

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$x = \pm \frac{a}{b}\sqrt{b^2 - y^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We see, from equation (1), that every value of x between $-a$ and $+a$ gives two real values of y, equal, with contrary signs; that the values of x, equal, respectively, to $-a$ and $+a$, give two values of y, each equal to 0; and that every value of x less than $-a$, and greater than $+a$, gives two imaginary values of y; this shows that the curve is limited in the direction of *positive* and *negative* abscissas, by two tangents, one at each vertex of the transverse axis; and that the curve is continuous between these limits, being symmetrical with respect to the transverse axis.

In like manner, we infer from equation (2) that the

curve is limited in the direction of *positive* and *negative* ordinates by two tangents, one at each vertex of the conjugate axis; and that the curve is continuous between these limits, being symmetrical with respect to the conjugate axis.

3°. *To find the parameter of the curve.*—The parameter is the breadth of the curve through either focus; that is, it is equal to the double ordinate through the focus. Making $x = \sqrt{a^2 - b^2}$, in [31], and denoting the corresponding value of $2y$ by $2p$, we have,

$$2p = \frac{2b^2}{a} = \frac{4b^2}{2a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Forming a proportion, from this equation, we have,

$$2a : 2b :: 2b : 2p \quad . \quad . \quad . \quad . \quad . \quad (4)$$

That is, *the parameter is a third proportional to the transverse and conjugate axes.*

4°. *To find the relation between the ordinates of any two points.*—Let (x', y') and (x'', y'') be two points. Substituting in [31], after solving with respect to y^2, we have the equations of condition that place the two points on the curve,

$$y'^2 = \frac{b^2}{a^2}(a^2 - x'^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$y''^2 = \frac{b^2}{a^2}(a^2 - x''^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Forming a proportion from (5) and (6), and factoring, we have,

$$y'^2 : y''^2 :: (a - x')(a + x') : (a - x'')(a + x'') \quad . \quad . \quad (7)$$

From which, we see, as in the case of the circle, (Art. **27**), that, *the squares of any two ordinates of the ellipse are to each other as the rectangles of the segments into which they divide the transverse axis.*

It is to be observed that the circle is but a particular case of the ellipse, *in which the axes are equal.* All the properties of the ellipse will, therefore, hold good for the circle. If $a = b$, the value of e reduces to 0; that is, the excentricity of the circle is 0; hence, the foci of the circle coincide at the centre. If we make $a = b$, in the equation of the ellipse, it becomes,

$$x^2 + y^2 = a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

which is the equation of a circle whose radius is a.

5°. *To determine the position of any point with respect to the curve.*—Let (x', y') be the point.

If the point is on the curve, its co-ordinates satisfy the equation of the curve, and we have,

$$a^2y'^2 + b^2x'^2 - a^2b^2 = 0 \quad . \quad . \quad . \quad . \qquad (9)$$

If the point is without the curve, draw a line from the centre to it, and designate the point in which it intersects the curve by P. It is obvious that the square of either co-ordinate of the given point is greater than the square of the corresponding co-ordinate of P; hence, in this case, we have,

$$a^2y'^2 + b^2x'^2 - a^2b^2 > 0 \quad . \quad . \quad . \quad . \qquad (10)$$

If the point is within the curve, draw a line from the centre to it, and prolong it till it intersects the curve, at a point P. The square of either co-ordinate of the

given point is less than the square of the corresponding co-ordinate of P; hence, in this case, we have,

$$a^2y'^2 + b^2x'^2 - a^2b^2 < 0 \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The expressions (9), (10), and (11), enable us to determine whether a given point lies *upon*, *without*, or *within* a given ellipse.

If we make $a = b = r$, in these expressions, we find corresponding expressions for the circle,

$$x'^2 + y'^2 - r^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$x'^2 + y'^2 - r^2 > 0 \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$x'^2 + y'^2 - r^2 < 0 \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Proposition 25.—*To find the equation of the ellipse referred to its principal vertex.*

48. Making $m = -a$ and $n = 0$, in [13] and [14], we have the following formulas for making the required transformation:

$$x = -a + x' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$y = y' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting in [31], we have,

$$a^2y'^2 + b^2(a^2 - 2ax' + x'^2) = a^2b^2 \quad . \quad . \quad . \quad (3)$$

Transposing, reducing, and dropping the dashes, we have,

$$y^2 = \frac{b^2}{a^2}(2ax - x^2) \quad . \quad . \quad . \quad . \quad . \quad [33]$$

which is, *the equation of the ellipse referred to the principal vertex.*

The axis of x is the transverse axis of the curve, and is an axis of symmetry; the axis of y is tangent to the curve at the principal vertex. Performing the indicated multiplication, substituting for $\frac{2b^2}{a}$ its value, $2p$, and denoting the square of the ratio of the semi-axes by r^2, we have,

$$y^2 = 2px - r^2x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad [34]$$

Proposition 26. — *To find the polar equation of an ellipse when the pole is at the right hand focus, and the initial line coincides with the transverse axis.*

49. The polar equation may be found by the method of Article **35**, but it is more readily deduced from equation (12), Article **46**. Assuming this equation, we have,

$$r = a - ex \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

From the figure of Article **46**, we have, since CF equals ae,

$$x = ae + r\cos\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting in (1), we find,

$$r = a - ae^2 - re\cos\phi \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Solving with respect to r, and factoring, we have,

$$r = \frac{a(1 - e^2)}{1 + e\cos\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad [35]$$

If we wish the pole to be at the left-hand focus, we use equation (11), Article **46**, in which case,

$$x = -ae + r'\cos\phi,$$

and the resulting polar equation is,

$$r' = \frac{a(1 - e^2)}{1 - e \cos \phi} \quad . \quad . \quad . \quad . \quad . \quad [36]$$

Discussion of the Equations.

50. If we make $\phi = 0$ in [35], we find,

$$r = a\frac{1 - e^2}{1 + e} = a(1 - e) = a - ae \quad . \quad . \quad (1)$$

If we make $\phi = 0$ in [36], we find,

$$r' = a\frac{1 - e^2}{1 - e} = a(1 + e) = a + ae \quad . \quad . \quad (2)$$

These values of r and r' are the distances from the foci to the right-hand vertex. By making $\phi = 180°$ we find in like manner,

$$r = a + ae,$$

and,

$$r' = a - ae \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

These values are the distances from the foci to the left-hand vertex. By making $\phi = 90°$, or $\phi = 270°$, we find,

$$r = a(1 - e^2),$$

and,

$$r' = a(1 - e^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

These are each equal to half the parameter of the curve, hence,

$$2p = 2a(1 - e^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

is another expression for the parameter. This expression

is identical with that deduced in Article **46**; for, it has been shown that,

$$b^2 = a^2(1 - e^2),$$

(Art. **46**), hence,

$$\frac{2b^2}{a} = 2a(1 - e^2).$$

Proposition 27. — *To find the polar equation when the pole is at the centre.*

51. Assume the equation of the curve referred to its centre,

$$a^2y^2 + b^2x^2 = a^2b^2 \quad . \quad . \quad . \quad . \quad . \quad [31]$$

Making m, n, and a, equal to 0 in [15] and [16], they become,

$$x = r\cos\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$y = r\sin\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting these in [31], we have,

$$a^2r^2\sin^2\phi + b^2r^2\cos^2\phi = a^2b^2.$$

Solving with respect to r, and taking the positive value only, we have,

$$r = \frac{ab}{\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}} \quad . \quad . \quad . \quad . \quad [37]$$

Discussion of the Equation.

52. If we make $\phi = 0$, or $\phi = 180°$ in [37], we find,

$$r = a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If we make $\phi = 90$, or $\phi = 270°$, we find,

$$r = b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These are the values of the semi-axes of the curve.

We know from trigonometry that,

$$\sin\phi = -\sin(180° + \phi),$$

and,

$$\cos\phi = -\cos(180° + \phi);$$

or,

$$\sin^2\phi = \sin^2(180° + \phi),$$

and,

$$\cos^2\phi = \cos^2(180° + \phi);$$

hence, the value of r is the same for any angle ϕ as for that angle increased by 180°; this shows that *every diameter of the ellipse is bisected at the centre.*

If we denote any semi-diameter of the ellipse by a', and its inclination to the axis by θ, we have,

$$a' = \frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}.$$

Squaring, and solving, with respect to the resulting denominator, we have,

$$a^2\sin^2\theta + b^2\cos^2\theta = \frac{a^2b^2}{a'^2} \quad . \quad . \quad . \quad . \quad (3)$$

In like manner denoting any other semi-diameter by b', and its inclination by θ', we have,

$$a^2\sin^2\theta' + b^2\cos^2\theta' = \frac{a^2b^2}{b'^2} \quad . \quad . \quad . \quad . \quad (4)$$

Formulas (3) and (4) are used in treating of the ellipse referred to conjugate diameters. If we substitute

for $\sin^2\phi$, its value, $1 - \cos^2\phi$, equation [37], becomes,

$$r = \frac{ab}{\sqrt{a^2 - (a^2 - b^2)\cos^2\phi}} \quad . \quad . \quad . \quad (5)$$

This value of r is *greatest* possible when its denominator is least, and the denominator is least when $\cos^2\phi$ is *greatest*. In like manner this value of r is least possible when $\cos^2\phi$ is *least*.

But, the greatest value of $\cos^2\phi$ is 1, which gives $r = a$; and the least value of $\cos^2\phi$ is 0, which gives $r = b$. Hence, we conclude that the transverse axis is the greatest, and the conjugate axis the least diameter of the ellipse.

Proposition 28.—*To find the equations of a tangent and normal to an ellipse.*

53. Assume the equation of an ellipse referred to its centre,

$$a^2y^2 + b^2x^2 = a^2b^2 \quad [31]$$

The equations of condition that place the points (x', y'), and (x'', y'') on the curve, are, (Art. **15**),

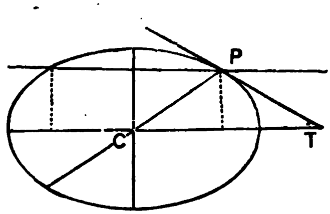

FIG. 31.

$$a^2y'^2 + b^2x'^2 = a^2b^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$a^2y''^2 + b^2x''^2 = a^2b^2 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Subtracting (1) from (2), transposing and factoring, we have,

$$a^2(y''-y')(y''+y') = -b^2(x''-x')(x''+x') \quad (3)$$

Dividing both members by $a^2(y''+y')(x''-x')$, we find,

$$\frac{y''-y'}{x''-x'} = -\frac{b^2(x''+x')}{a^2(y''+y')} \quad . \quad . \quad . \quad . \quad (4)$$

The second member of (4) is the slope of the secant, (Art. **29**); if we make $y''=y'$ and $x''=x'$, we find for the slope of the tangent,

$$\tan\theta' = -\frac{b^2x'}{a^2y'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Substituting this value in [23], we have the equation of the tangent,

$$y-y' = -\frac{b^2x'}{a^2y'}(x-x') \quad . \quad . \quad . \quad . \quad (6)$$

Reducing, transposing, and substituting for $a^2y'^2+b^2x'^2$ its value from equation (1), we have,

$$a^2yy'+b^2xx' = a^2b^2 \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which is the equation of the tangent.

Substituting in [24], we have the equation of the normal,

$$y-y' = \frac{a^2y'}{b^2x'}(x-x') \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Substituting in [25] and [26], we have the values of the subtangent and subnormal,

$$\text{S.T} = -\frac{a^2y'^2}{b^2x'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\text{S.N} = -\frac{b^2x'}{a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If we make $y=0$ in (7), and find the corresponding value of x, we have,

$$x=\frac{a^2}{x'} \quad . \quad . \quad (11)$$

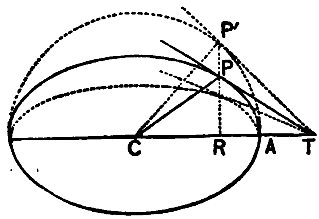

FIG. 32.

But, x, in this case, denotes the distance CT; x' the distance CR; and a the distance CA; hence,

$$CR : CA :: CA : CT \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If we make $a=b=r$, in (7), (8), (9), and (10), we find the corresponding expressions for the circle, which are identical with those found in Article **30.**

If we substitute in (9) the value of $\frac{a^2y'^2}{b^2}$, deduced from (1), we find,

$$S.T=\frac{a^2-x'^2}{x'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

This expression for the subtangent is independent of both b and y'. Hence, if any number of ellipses be constructed on a given transverse axis, and tangents be drawn to them at the points corresponding to any given abscissa, these tangents will all intersect the axis of x at a common point. This group of ellipses includes the circle whose diameter is the given transverse axis.

This principle enables us to draw a tangent to an ellipse at a given point.

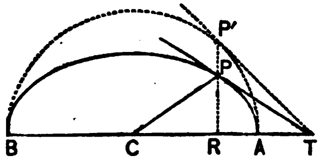

FIG. 33.

Let APB be an ellipse, and P any point of the curve; on AB, as a diam-

eter, describe a semicircle; draw the ordinate of P, and produce it to intersect the semicircle at P′; draw the tangent P′T to the circle, and draw also the line PT; this will be the required tangent, because TR is the common subtangent of the tangents to the circle and ellipse.

If we substitute $(-x', -y')$ for (x', y') in equation (5), the value of $\tan\theta'$ will be unchanged; but this substitution is equivalent to changing the point of contact from the *upper* to the *lower vertex* of the same diameter; hence, the two tangents to the ellipse at the opposite extremities of any diameter are parallel to each other.

Supplementary Chords and Conjugate Diameters.

54. Two chords are *supplementary*, when they are drawn from the vertices of any diameter to the same point of the curve.

Two diameters are *conjugate*, when one is parallel to the tangents at the vertices of the other.

Let it be required to find the equation of condition for supplementary chords, drawn from the vertices of the transverse axis.

If we replace a by $\tan\theta''$, and make,

$$x' = -a \quad \text{and} \quad y' = 0,$$

in equation [4], we have, for the equation of any chord through the left hand vertex of the transverse axis,

$$y = \tan\theta''(x + a) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In like manner, replacing a by $\tan\theta'''$, and making $x' = +a$ and $y' = 0$, we have, for the equation of any chord through the right hand vertex of the transverse axis,

$$y = \tan\theta''' (x - a) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These lines must intersect, because their slopes are different. Combining (1) and (2) by multiplication, we have,

$$y^2 = \tan\theta'' \tan\theta''' (x^2 - a^2) \quad . \quad . \quad . \quad . \quad (3)$$

in which x and y are the co-ordinates of the point of intersection. In order that this point may lie on the ellipse, the values of x and y in (3) must be such as to satisfy the equation of the ellipse, which can be written under the form,

$$y^2 = -\frac{b^2}{a^2}(x^2 - a^2) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Making (3) and (4) simultaneous, by equating the values of y^2, and then suppressing the common factor, $x^2 - a^2$, we have,

$$\tan\theta'' \tan\theta''' = -\frac{b^2}{a^2} \quad . \quad . \quad . \quad . \quad . \quad [38]$$

which is *the equation of condition for supplementary chords.*

To find the equation of condition for conjugate diameters, we have, for the slope of a tangent at the point (x', y'), and consequently for the slope of a diameter parallel to the tangent, (Art. **53**),

$$\tan\theta' = -\frac{b^2x'}{a^2y'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Denoting the inclination of the diameter drawn to the point (x', y') by θ, we have, from the figure,

$$\tan\theta = \frac{y'}{x'} \quad . \quad . \quad (6)$$

Multiplying (5) and (6), member by member, we have,

$$\tan\theta\tan\theta' = -\frac{b^2}{a^2} \quad . \quad . \quad [39]$$

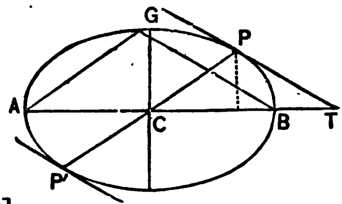

FIG. 34.

which is *the equation of condition for conjugate diameters.* It is also *the equation of condition for a diameter and a tangent at its vertex.*

If we make, $\tan\theta = \frac{\sin\theta}{\cos\theta}$, and $\tan\theta' = \frac{\sin\theta'}{\cos\theta'}$, in [39], and then clear of fractions and transpose, we have,

$$a^2\sin\theta\sin\theta' + b^2\cos\theta\cos\theta' = 0 \quad . \quad . \quad [40]$$

which is another form of equation [39], used in treating of conjugate diameters.

If we assume any value for $\tan\theta''$ in [38], or for $\tan\theta$ in [39], we find from the equations, corresponding values for $\tan\theta'''$ and $\tan\theta'$. Hence, *every chord has a supplement, and every diameter a conjugate.*

If we make $\theta = 0$, we have $\sin\theta = 0$, and $\cos\theta = 1$. Substituting these in [40], we find $\cos\theta' = 0$, or $\theta' = 90$. This shows that the axes are conjugate diameters.

The second members of [38] and [39] are equal; hence,

$$\tan\theta''\tan\theta''' = \tan\theta\tan\theta' \quad . \quad . \quad . \quad . \quad (7)$$

If $\tan\theta = \tan\theta''$, we have $\tan\theta' = \tan\theta'''$, and the reverse.

Hence, if a diameter is parallel to a chord through the principal vertex, the conjugate of the diameter is parallel to the supplement of the chord, and the reverse.

Also, if a diameter is parallel to a chord through the principal vertex, the tangents at the vertices of the diameter are parallel to the supplement of the chord, and the reverse.

These principles form the basis of the following constructions:

1°. *To draw a tangent to an ellipse at a given point.*—Let GBA be the ellipse, AB its transverse axis, and P the given point. Draw the diameter PP′, and through A draw the chord AG parallel to it; draw the supplement, GB, of the given chord; and through P draw PT parallel to GB; this line is the required tangent.

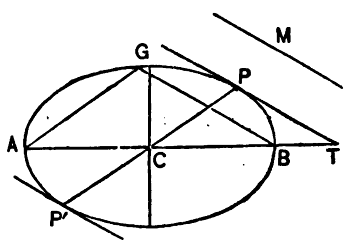

FIG. 35.

2°. *To draw a tangent to an ellipse parallel to a given line.*—Let M be the given line; draw BG parallel to M, and draw its supplement, GA; then draw the diameter, P′P, parallel to AG, and through P draw PT parallel to GB; this is the required tangent.

Proposition 29.—*To prove that a normal bisects the angle between the focal lines to the point of contact.*

55. If we make $y = 0$, in the equation of the normal, (Equation (8), Art. **53**), we find for the corresponding value of x, which denotes the distance CN,

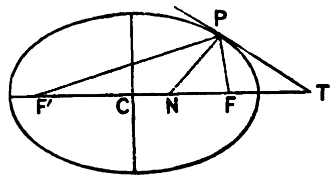

FIG. 36.

$$x = \frac{a^2 - b^2}{a^2} x' = e^2 x'.$$

Adding this to, and subtracting it from, ae, we find,

$$F'N = e(a + ex') \text{ and } FN = e(a - ex') \quad . \quad . \quad (1)$$

From equations (11) and (12), Art. **46**, we have for the focal lines to the point (x', y'),

$$r' = a + ex' \text{ and } r = a - ex' \quad . \quad . \quad . \quad (2)$$

Hence, F′N and FN are equimultiples of r' and r; consequently, we have,

$$r' : r :: F'N : FN \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The normal therefore divides the base of the triangle F′PF into segments proportional to the adjacent focal lines; hence, it bisects the angle between these lines, *which was to be proved.*

The tangent is perpendicular to the normal; consequently, it makes equal angles with the focal lines to the point of contact. This principle gives rise to the following constructions:

1°. *To draw a tangent to an ellipse at a given point.*—Let F′ and F be the foci, P the given point, and F′P, FP, focal lines to the point of contact. Prolong F′P, making PK equal to PF, and join the points K and F; through P draw a line perpendicular to FK, and it will be the required tangent; for, it will bisect the angle FPK, and consequently, will make equal angles with the focal lines to the point of contact.

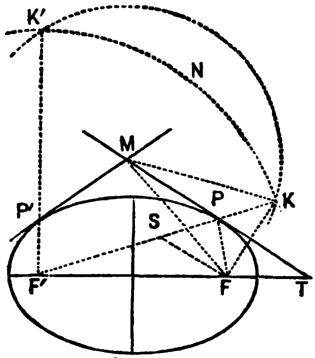

FIG. 37.

If a distance equal to PF be laid off from P towards F′, and its extremity be joined with F, the resulting line will be parallel to the tangent; and conversely, a line through P, parallel to the resulting line, will be a tangent to the curve at P.

2°. *To draw a tangent to an ellipse through a given point.*—Let M be the given point. From one focus, F′, as a centre, with a radius equal to the transverse axis, describe an arc K′NK; from the given point, with a radius equal to its distance from the other focus F, describe a second arc, cutting the first in K and K′; draw the line F′K, cutting the ellipse at P, and then draw the line MP; this line willl be tangent to the ellipse at P; for, PK equals PF, because each, added to F′P, will give the transverse axis; and MK equals MF, because they are radii of the same circle; hence, MP is

perpendicular to FK at its middle point; consequently, for the reason given in the last construction, is tangent to the ellipse at P.

A second tangent, MP′, may be found by drawing the line F′K′, and then uniting the point in which it cuts the ellipse with the given point.

It may be shown from the figure that there will be *two* tangents, if M is without the curve, but *one*, if M is on the curve, and none at all, if M is within the curve.

Proposition 30.—*To find the relation between the ordinates of an ellipse and its circumscribing circle.*

56. If a circle be described on the transverse axis of an ellipse as a diameter, the two curves will have the vertices of that diameter in common; but from what was shown in Article **52**, all other points of the ellipse will lie within the circle; hence, the circle is said to *circumscribe* the ellipse.

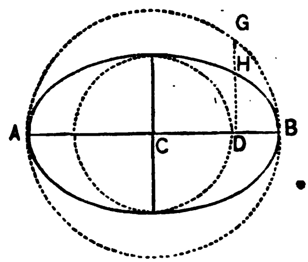

FIG. 38.

Any ordinate of the circle, as DG, will intersect the ellipse at some point, H; the points G and H are called *corresponding points*, and the ordinates, DG and DH, are called *corresponding ordinates*.

The equation of the ellipse when referred to its centre may be written under the form,

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If we denote the ordinates of the circle by y', to distinguish them from the ordinates of the ellipse, the equation of the circumscribing circle may be placed under the form,

$$y'^2 = a^2 - x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If we make the values of x equal, in (1) and (2), the values of y and y' will represent corresponding ordinates. Dividing (1) by (2), member by member, we have,

$$\frac{y^2}{y'^2} = \frac{b^2}{a^2},$$

or,

$$\frac{y}{y'} = \frac{b}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Forming a proportion from (3), we have,

$$y' : y :: a : b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

That is, *any ordinate of the circumscribing circle is to the corresponding ordinate of the ellipse, as the semi-transverse, is to the semi-conjugate axis.*

If a circle be described on the conjugate axis as a diameter, it is said to be *inscribed* in the ellipse. In this case the *corresponding points* are those that have equal ordinates; the abscissas of these points are *corresponding abscissas.* It may be shown, as before, that *any abscissa of the inscribed circle is to the correspond-*

ing abscissa of the ellipse, as the semi-conjugate, is to the semi-transverse axis.

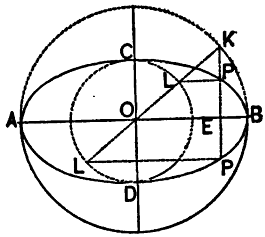

FIG. 39.

The principle demonstrated in this article enables us to construct the curve by points. Let AB and CD be the axes of an ellipse. On these, as diameters, describe two circles; at any point of AB, as E, erect a perpendicular, EK, and join K with O; through the point L, in which this line intersects the smaller circle, draw a parallel to AB, cutting EK in P; then is P a point of the ellipse.

For, we have,

$$EK : EP :: OK : OL,$$

or,

$$y' : EP :: a : b \quad . \; . \; . \; . \; . \; . \; . \quad (5)$$

Comparing (4) and (5), we see that $EP = y$, hence, P, is on the curve. In like manner any number of points may be found.

The angle BOK is called the *excentric angle of the point* P. Denoting this angle by ϕ, we find from the figure,

$$\left.\begin{aligned} x &= a\cos\phi, \\ y' &= a\sin\phi \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (6)$$

and,

But, EP being equal to y, we have from (5),

$$y' = \frac{a}{b}y \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

Substituting in (6), and reducing, we have,

$$\text{and,} \qquad \left.\begin{aligned} x &= a\cos\phi, \\ y &= b\sin\phi \end{aligned}\right\} \quad \ldots\ldots\ldots (8)$$

Equations (8) are formulas for the co-ordinates of any point of an ellipse, in terms of the axes and the excentric angle of that point.

Proposition 31. — *To find the equation of an ellipse referred to any pair of conjugate diameters.*

57. Assume the equation of the ellipse referred to its centre and axes,

$$a^2y^2 + b^2x^2 = a^2b^2 \quad \ldots\ldots \quad [31]$$

Denote the inclination of any diameter by θ, and the inclination of its conjugate by θ'. Making $m = 0$, $n = 0$, $a = \theta$, and $a' = \theta'$, in [9] and [10], we have the following formulas, for making this particular transformation:

$$x = x'\cos\theta + y'\cos\theta' \quad \ldots\ldots \quad (1)$$

$$y = x'\sin\theta + y'\sin\theta' \quad \ldots\ldots \quad (2)$$

Squaring, substituting in [31], and arranging the resulting equation, we have,

$$\left.\begin{aligned} &a^2\sin^2\theta' \\ &+ b^2\cos^2\theta' \end{aligned}\right| y'^2 + \left.\begin{aligned} &a^2\sin^2\theta \\ &+ b^2\cos^2\theta \end{aligned}\right| x'^2 + \left.\begin{aligned} &a^2\sin\theta\sin\theta' \\ &+ b^2\cos\theta\cos\theta' \end{aligned}\right| 2x'y' = a^2b^2 \,. \ (3)$$

But, from equation [40], we see that the coefficient of $2x'y'$ is equal to 0, and from equations (3), (4), of Article **52**, we see that the coefficients of x'^2 and y'^2 are

respectively equal to $\frac{a^2b^2}{a'^2}$ and $\frac{a^2b^2}{b'^2}$. Making these substitutions in (3), and suppressing the common factor a^2b^2, we have,

$$\frac{y'^2}{b'^2}+\frac{x'^2}{a'^2}=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Dropping the accents from x and y, because they are general variables, and clearing of fractions, we have,

$$a'^2y^2+b'^2x^2=a'^2b'^2 \quad . \quad . \quad . \quad . \quad . \quad [41]$$

which is the required equation.

Equation [41], is of the same form as [31], as it should be, since the axes are conjugate diameters. Equation [41] may be regarded as the general equation, of which [31] is a particular case.

Discussion of the Equation.

58. Equation [41] being of the same form as equation [31], it follows that all the analytical conclusions deducible from the latter, are deducible from the former; the only difference in the two cases lies in the interpretation of the results.

By proceeding as in Article **47**, we may deduce the following properties:

1°. The curve cuts the axis of x in the points, $(-a', 0)$ and $(+a', 0)$, and the axis of y in the points, $(0, -b')$ and $(0, +b')$.

2°. The curve is limited by two pairs of parallel tan-

gents, one pair drawn at the vertices of each of the conjugate diameters.

3°. The curve is obliquely symmetrical with respect to both conjugate diameters; that is, each bisects all the chords parallel to the other.

Inasmuch as the axis of x may be taken to coincide with any diameter of the ellipse, it follows that every diameter of the ellipse bisects all the chords parallel to the conjugate of that diameter.

4°. The squares of any two ordinates to either diameter, are proportional to the rectangles of the segments into which they divide that diameter.

5°. Any point, (x', y'), referred to the same diameters as the curve, will lie *without*, *upon*, or *within* the curve, according as,

$$a'^2y'^2 + b'^2x'^2 - a'^2b'^2,$$

is *greater than*, *equal to*, or *less than* 0.

Properties and Relations of Conjugate Diameters.

59. Let BHA be an ellipse; BGA its circumscribed circle, and let HK and H'K' be any pair of conjugate diameters.

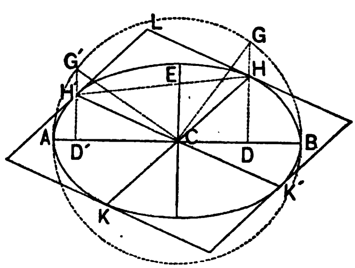

FIG. 40.

Let the ordinates DH and D'H' be prolonged to meet the circle at

the corresponding points, G and G′. Denote the excentric angle of H by ϕ, and that of H′ by ϕ', and let inclinations of HK and H′K′ be denoted by θ and θ'.

The co-ordinates of H, denoted by x and y, as given in equations (8), Article **56**, are,

and, $$\left.\begin{aligned} x &= a\cos\phi, \\ y &= b\sin\phi \end{aligned}\right\} \quad \ldots \ldots \quad (1)$$

In like manner, the co-ordinates of H′, denoted by x' and y', are given by the equations,

and, $$\left.\begin{aligned} x' &= a\cos\phi', \\ y' &= b\sin\phi' \end{aligned}\right\} \quad \ldots \ldots \quad (2)$$

In equations (1) and (2), $\cos\phi$ and $\cos\phi'$, have the same signs as x and x'.

From the figure, we have,

and, $$\left.\begin{aligned} \tan\theta &= \frac{y}{x}, \\ \tan\theta' &= \frac{y'}{x'} \end{aligned}\right\} \quad \ldots \ldots \quad (3)$$

In equations (3), $\tan\theta$ and $\tan\theta'$ have the same signs as x and x'.

Substituting the values of x, y, x', and y', taken from (1) and (2), in equations (3), we have,

and, $$\left.\begin{aligned} \tan\theta &= \frac{b}{a}\tan\phi, \\ \tan\theta' &= \frac{b}{a}\tan\phi' \end{aligned}\right\} \quad \ldots \ldots \quad (4)$$

In equations (4), $\tan\theta$, has the same sign as $\tan\phi$, and $\tan\theta'$, the same sign as $\tan\phi'$

Multiplying equations (4), member by member, we have,

$$\tan\theta\tan\theta' = \frac{b^2}{a^2}\tan\phi\tan\phi' \quad . \quad . \quad . \quad . \quad (5)$$

But, from Article **54**, we have,

$$\tan\theta\tan\theta' = -\frac{b^2}{a^2} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Combining (5) and (6), and reducing, we have,

$$\tan\phi\tan\phi' + 1 = 0\,; \quad \therefore \quad \phi' = 90^\circ + \phi \quad . \quad . \quad (7)$$

That is, *the excentric angles of the upper vertices of two conjugate diameters differ from each other by* 90°. Substituting for ϕ', its value $90^\circ + \phi$, in (2), and reducing, we have,

and,
$$\left.\begin{aligned} x' &= -a\sin\phi, \\ y' &= b\cos\phi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Squaring both members of (1), and adding, we find for $\overline{\mathrm{CH}}^2$, denoted by a'^2, the expression,

$$a'^2 = a^2\cos^2\phi + b^2\sin^2\phi \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In like manner, from (8), we find for b'^2, the expression,

$$b'^2 = a^2\sin^2\phi + b^2\cos^2\phi \quad . \quad . \quad . \quad . \quad (10)$$

Adding (9) and (10), member to member, and substituting 1 for $\sin^2\phi + \cos^2\phi$, we have,

$$a'^2 + b'^2 = a^2 + b^2 \quad . \quad . \quad . \quad . \quad . \quad (11)$$

That is, *the sum of the squares of any pair of semi-*

conjugate diameters, is equal to the sum of the squares of the semi-axes.

The area of the triangle HCH′, is equal to the area of the trapezoid HDD′H′, diminished by the sum of the areas of the triangles CDH and CD′H′. Hence, denoting the area of the triangle by T, we have, from the principles of mensuration,

$$\mathrm{T} = \frac{y + y'}{2}(x + x') - \frac{xy}{2} - \frac{x'y'}{2} = \frac{xy' + x'y}{2} \quad . \; . \quad (12)$$

But, twice the triangle T, is equal to the parallelogram CH′LH, described on a' and b'. Multiplying both members of (12) by 2, and denoting the parallelogram CL by P, we have,

$$\mathrm{P} = xy' + x'y \quad . \; . \; . \; . \; . \; . \quad (13)$$

In this equation, we have used the *numerical* value of x', without regard to its sign. Substituting for x and y their values taken from (1), and for x' and y' their values taken from (8), (disregarding the sign of x'), we have,

$$\mathrm{P} = ab\,(\cos^2\phi + \sin^2\phi) = ab \quad . \; . \; . \quad (14)$$

That is, *the parallelogram on any pair of semi-conjugate diameters, is equal to the parallelogram on the semi-axes.*

Proposition 32.—*To find the equation of a tangent referred to any pair of conjugate diameters.*

60. The equation of a straight line through two points, when referred to oblique axes is, (Art. **23**),

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \quad . \; . \; . \; . \quad [17b]$$

in which $\frac{y''-y'}{x''-x'}$, is the ratio of the sines of the angles that the line makes with the two axes.

If the two points lie on the ellipse, we find for the coefficient of the second member, as in Article **53**,

$$\frac{y''-y'}{x''-x'} = -\frac{b'^2(x''+x')}{a'^2(y''+y')} \quad . \quad . \quad . \quad . \quad (1)$$

If we suppose the *second* point to coincide with the *first*, this coefficient becomes $-\frac{b'^2x'}{a'^2y'}$, and this substituted in [17*b*], gives the required equation,

$$y-y' = -\frac{a'^2x'}{b'^2y'}(x-x') \quad . \quad . \quad . \quad . \quad (2)$$

which may, as in Article **53**, be reduced to the form,

$$a'^2yy' + b'^2xx' = a'^2b'^2 \quad . \quad . \quad . \quad . \quad . \quad [42]$$

Equation [42], *is the required equation.*

If we make $y=0$, in [42], we find for the corresponding value of x,

$$x = \frac{a'^2}{x'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In this case, x is the distance from the centre to the point in which the tangent intersects the axis of x, and x' is the abscissa of the point of contact. Equation (3) shows that,

$$x' : a' :: a' : x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

a property analogous to that shown in Article **53**.

Of Poles and Polars.

61. Let (m, n) be a point referred to the same axes as the ellipse, in Article **57**, and assume the equation of a tangent to the curve, also referred to the same axes,

$$a'^2yy' + b'^2xx' = a'^2b'^2 \quad . \quad . \quad . \quad . \quad . \quad [42]$$

The equation of condition that causes this tangent to pass through the point (m, n) is, (Art. **15**),

$$a'^2ny' + b'^2mx' = a'^2b'^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But it has been shown, (Art. **55**), that two tangents can always be drawn to an ellipse from a point without the curve; hence, there are two sets of values of x' and y' that will satisfy equation (1), and at the same time satisfy the equation of the ellipse. The quantities x' and y', in equation (1), are therefore the co-ordinates of both points of contact.

Equation (1) is obviously the equation of condition that places the two points of contact on the line whose equation is,

$$a'^2ny + b'^2mx = a'^2b'^2 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Because equation (2) is satisfied by the co-ordinates of both points of contact, it must be the equation of the straight line passing through those points; that is, *it is the equation of the chord of contact.*

If we make $y = 0$, in equation (2), we find,

$$x = \frac{a'^2}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This is the abscissa of the point in which the chord of contact intersects the axis of x, and this abscissa re-

mains constant so long as m is unchanged. If, therefore, we suppose m to be fixed in value, whilst n takes in succession every value from $-\infty$ to $+\infty$, the point (m, n) will generate a line parallel to the axis of y, such that, if two tangents be drawn from any point of it to the ellipse, the corresponding chord of contact will pass through the point on the axis of x, whose abscissa is given by equation (3).

The line from which the tangents are drawn is called the *polar* of the point; and the point in which the chords of contact intersect is called the *pole* of the line.

The *polar point* of the line is the point in which the polar cuts the axis of x; that is, it is the point on the axis of x whose abscissa is m. We see that m and x can change places in equation (3); hence, if the *primitive polar point* be taken as a *pole*, the *primitive pole* will be *the new polar point.*

It is obvious that every straight line in the plane of the curve has a corresponding *pole*, and that every point in the plane has a corresponding *polar.*

From the principles already explained, we deduce the following constructions.

To find the pole of a given line taken as a polar.— Draw a semi-diameter parallel to the polar, and find its semi-conjugate; produce the latter (if necessary) till it intersects the polar; the point of intersection will be the *polar point;* then construct a third proportional to the distance of this point from the centre, and the semi-diameter last found; the extremity of this distance will be the required *pole,* (Equation (3)).

To find the polar of a given point taken as a pole.—Draw a diameter through the given pole, and prolong it indefinitely; also find the conjugate of this diameter; then construct a third proportional to the distance from the centre to the pole, and the first semi-diameter; the extremity of this distance will be the *polar point;* through this, draw a line parallel to the second diameter, and it will be the required *polar.*

If the *pole* is within the curve, the corresponding *polar point* is without, and the reverse.

Hence, if the pole is within the ellipse, the corresponding *polar* does not intersect the curve; if the point is without the ellipse, the corresponding polar cuts the curve in two points; and if the point is on the ellipse, the corresponding polar is tangent to the curve at that point.

The *polar of the centre* is at an infinite distance, and the *pole of a diameter* is at an infinite distance from the centre.

The polar of either focus is a *directrix* of the curve; that is, it is a line such that the distance of any point of the curve from it, bears a *constant ratio* to the focal distance of the same point. For, making $CR = ae$, and $CA = a$, in proportion (12), Art. **53**, we have,

$$CT = \frac{a^2}{ae} = \frac{a}{e} \quad \ldots\ldots \quad (4)$$

Hence, the distance from any point whose abscissa is x, to the perpendicular through T, denoted by d, is given by the equation,

$$d = \frac{a}{e} - x = \frac{a - ex}{e} \quad \ldots\ldots \quad (5)$$

But, from equation (12), Art. **46**, we have for the focal distance of the same point,

$$r = a - ex \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence, we have,

$$ed = r, \quad \text{or,} \quad \frac{r}{d} = e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which was to be shown.

The constant ratio is the *excentricity* of the ellipse.

PROBLEMS.

1°. Find the semi-axes of the ellipse,

$$3y^2 + 2x^2 = 6.$$

Solution.—Dividing both members by 6, we have,

$$\frac{y^2}{2} + \frac{x^2}{3} = 1.$$

Comparing this with equation [32], we have,

$$a = \sqrt{3} \quad \text{and} \quad b = \sqrt{2}. \quad \textit{Ans.}$$

2°. Find the semi-axes of the ellipse,

$$4y^2 + 3x^2 = 19.$$

$$\textit{Ans.} \quad a = \sqrt{\frac{19}{3}} \quad \text{and} \quad b = \sqrt{\frac{19}{4}}.$$

3°. Find the points of intersection of the straight line, $y = x + 1$, and the ellipse, $2y^2 + x^2 = \frac{11}{3}$.

$$\textit{Ans.} \quad \left(\frac{1}{3}, \frac{4}{3}\right) \quad \text{and} \quad \left(-\frac{5}{3}, -\frac{2}{3}\right).$$

4°. Find the points of intersection of the parabola, $y^2 = 4x$, and the ellipse, $3y^2 + 2x^2 = 14$.

Ans. (1, 2) and (1, − 2).

5°. Find the value of that radius-vector, which is inclined 60° to the transverse axis in the ellipse, $25y^2 + 16x^2 = 400$, the pole being at the right hand focus.

Ans. $\frac{32}{13}$.

6°. Find the equation of a tangent to the ellipse, $3y^2 + 2x^2 = 35$, at the point whose abscissa is 2.

Ans. $\frac{105}{2}y + \frac{70}{3}x = \frac{1225}{6}$, or, $9y + 4x = 35$.

7°. Find the inclination of a tangent to the ellipse, $25y^2 + 9x^2 = 225$, at the point whose abscissa is 2.

Ans. 165° 20′.

8°. Find a formula for the distance from the point (x', y') to any point of the ellipse, $a^2y^2 + b^2x^2 = a^2b^2$.

Solution.—From formula [1], we have,

$$d = \sqrt{(x'-x)^2 + (y'-y)^2} = \sqrt{(x'-x)^2 + \left(y' - \frac{b}{a}\sqrt{a^2-x^2}\right)^2};$$

or developing,

$$d = \sqrt{x'^2 - 2xx' + x^2 + y'^2 - \frac{2by'}{a}\sqrt{a^2 - x^2} + \frac{b^2}{a^2}(a^2 - x^2)}.$$

Ans.

9°. Find the condition that will make the value of d, in the last problem, rational in terms of x.

Solution.—The subordinate radical must disappear; this can only happen by supposing,

$$y' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Making this supposition, we have for d,

$$d = \sqrt{(x'^2 + b^2) - 2xx' + \frac{a^2 - b^2}{a^2} x^2}.$$

In order that d may be rational in terms of x, we must have,

$$-2xx' = 2\sqrt{x'^2 + b^2} \times \sqrt{a^2 - b^2} \times \frac{x}{a}.$$

Dividing by $2x$ and squaring both members, we have,

$$x'^2 = (x'^2 + b^2)\frac{a^2 - b^2}{a^2}.$$

whence,

$$x'^2\left(1 - \frac{a^2 - b^2}{a^2}\right) = \frac{b^2(a^2 - b^2)}{a^2};$$

or,

$$x'^2 = a^2 - b^2; \quad \therefore \quad x' = \pm\sqrt{a^2 - b^2} \quad . \quad . \quad (2)$$

From equations (1) and (2) we infer that the required condition is, that the point (x', y') shall coincide with one of the foci.

10°. Find the equation of a tangent to the ellipse in terms of its slope and semi-axes.

Solution.—Solving equation (7), Article **53**, with reference to y, we have,

$$y = -\frac{b^2 x'}{a^2 y'} x + \frac{b^2}{y'} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Denoting the slope of the tangent by $\tan\theta$, we have,

$$\tan\theta = -\frac{b^2x'}{a^2y'}; \quad \therefore \quad bx' = -\frac{a^2y'\tan\theta}{b}.$$

Squaring both members,

$$b^2x'^2 = \frac{a^4y'^2\tan^2\theta}{b^2}.$$

Substituting this in the equation,

$$a^2y'^2 + b^2x'^2 = a^2b^2,$$

we find,

$$a^2y'^2 + \frac{a^4y'^2}{b^2}\tan^2\theta = a^2b^2.$$

Dividing by $\frac{a^2}{b^2}$, and factoring, we have,

$$y'^2(b^2 + a^2\tan^2\theta) = b^4;$$

whence, by extracting the square root of both members,

$$\frac{b^2}{y'} = \sqrt{b^2 + a^2\tan^2\theta}.$$

Substituting in (1), we have,

$$y = x\tan\theta + \sqrt{b^2 + a^2\tan^2\theta} \quad . \quad . \quad . \quad (2)$$

which is the required equation.

11°. Find the *locus* of the intersection of any tangent and a perpendicular to it, from the focus.

Solution.—The equation of a straight line through the focus perpendicular to the tangent is,

$$y = -\frac{1}{\tan\theta}(x - ae) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which ae is the abscissa of the focus.

If we make equation (1) simultaneous with equation (2) of the last problem, x and y will be the co-ordinates of the point of intersection.

From (1) we have,

$$y \tan\theta + x = ae \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (2) we have,

$$y - x\tan\theta = \sqrt{b^2 + a^2\tan^2\theta} \quad . \quad . \quad . \quad . \quad (3)$$

Squaring (2) and (3), and adding, member to member, and recollecting that $b^2 + a^2e^2 = a^2$, we have,

$$(x^2 + y^2)(1 + \tan^2\theta) = a^2(1 + \tan^2\theta);$$

whence, by division,

$$x^2 + y^2 = a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence, *the required locus is the circumscribed circle.*

12°. Find the excentricity of the ellipse,

$$3y^2 + 2x^2 = d^2,$$

d being any number.

Ans. $e = \sqrt{\frac{1}{3}}$.

13°. Find a point on the ellipse,

$$3y^2 + 2x^2 = 12,$$

at which the tangent is equally inclined to both axes.

Ans. $\left(3\sqrt{\frac{2}{5}},\ 2\sqrt{\frac{2}{5}}\right)$.

VIII. Of the Hyperbola.

Definitions of Terms.

62. An **Hyperbola** is a plane curve, that may be generated by a point, moving so that the *difference* of its distances from two *fixed points* is equal to a *given line.*

The fixed points are called *foci.* The *transverse axis,* is a straight line through the foci, limited by the curve. The *centre,* of the hyperbola is that point of the transverse axis which is midway between the foci. The *conjugate axis,* is a straight line through the centre, perpendicular to the transverse axis.

Since the conjugate axis is perpendicular to the line joining the foci, at its middle point, every point of it must be equally distant from the foci; hence, from the law of generation there can be no point of the curve on this line; that is, the conjugate axis does not intersect the curve: but there are points on each side of the conjugate axis; hence, the curve is made up of two parts, called *branches.* The length of the conjugate axis is such, that the diagonal of the rectangle described on it and the transverse axis, is equal to the distance between the foci.

The *excentricity,* is the distance from the centre to either focus, divided by the semi-transverse axis.

A *diameter,* is a straight line passing through the centre. When it cuts the curve the points of intersection are called *vertices.* The right-hand vertex of the transverse axis, is called the *principal vertex* of the curve.

Construction of the Curve.

63. The curve can be constructed by *continuous movement*, or *by points*.

1°. *By continuous movement.* Let F and F′ be the foci; fasten a ruler, F′H, at F′, so that it can revolve about F′ as a centre; take a string whose length is less than that of the ruler by the *given distance*, and fasten one end at F and the other end at H; press the string against the ruler by a pencil, P, and revolve the ruler around F′; the point P will describe one branch of the hyperbola; for in every position of P, we have PF′ − PF equal to the given distance. By using a string whose length exceeds that of the ruler by the given distance, we may in like manner describe the other branch. It may be shown, as in Article **45**, that the *given distance* is equal to the transverse axis.

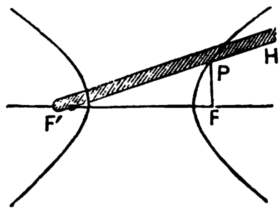

FIG. 41.

2°. *By points.* With a distance greater than F′B as a radius, and from F′ as a centre, describe an arc; with a radius equal to that before employed, diminished by the transverse axis, AB, and from F as a centre,

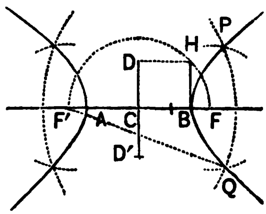

FIG. 42.

describe a second arc, cutting the first at P and Q; these points are points of the curve: for, each satisfies the definition of a point of the curve. By using the same radii, and reversing the centres, two points of the other branch may be found. When a sufficient number of points are found, draw a curve through them, and it will be the required curve.

3°. *To find the length of the conjugate axis.* Describe a semicircle on F′F as a diameter; from the principal vertex of the curve, B, draw the ordinate BH; through H draw HD, parallel to BA till it meets DD′ in D; CD is the semi-conjugate axis. For, CH, the semi-diagonal of the parallelogram described on AB and DD′ is equal to CF.

Proposition 33.—*To find the equation of the hyperbola referred to its centre.*

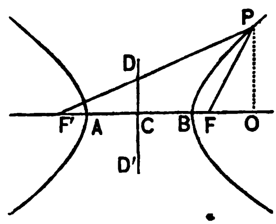

FIG. 43.

64. Let C be the centre, AB and D′D the axes, and P any point of the hyperbola. Denote the focal lines, FP and F′P, by r and r', the semi-transverse axis by a, the semi-conjugate axis by b, the distance, CF or CF′, by c, and the co-ordinates of P by x and y. Then will FO be equal to $x - c$, and F′O to $x + c$.

From the right-angled triangles FOP and F′OP, we have,

$$r^2 = y^2 + (x - c)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$r'^2 = y^2 + (x + c)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From the definition of the hyperbola, we have,

$$r' - r = 2a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Adding (1) and (2), and subtracting (1) from (2), we have,

$$r'^2 + r^2 = 2(y^2 + x^2 + c^2) \quad . \quad . \quad . \quad . \quad (4)$$

$$r'^2 - r^2 = 4cx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Factoring (5), and dividing by (3), member by member, we have,

$$r' + r = \frac{2cx}{a} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Combining (3) and (6), we find for the focal lines,

$$r' = + a + \frac{cx}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$r = - a + \frac{cx}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Squaring (7) and (8), substituting in (4), and reducing, we have,

$$a^2 + \frac{c^2x^2}{a^2} = y^2 + x^2 + c^2 \quad . \quad . \quad . \quad . \quad (9)$$

But,

$$c^2 = b^2 + a^2,$$

from the definition of the length of the conjugate axis, (Art. **62**); hence, equation (9) reduces to

$$a^2 + \frac{b^2 + a^2}{a^2}x^2 = y^2 + x^2 + a^2 + b^2 \quad . \quad . \quad (10)$$

Transposing and reducing, we have,

$$a^2y^2 - b^2x^2 = - a^2b^2 \quad . \quad . \quad . \quad . \quad . \quad [43]$$

This equation is true for every position of the point P; hence, *it is the required equation.*

Dividing both members of [43] by a^2b^2, we have,

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=-1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [44]$$

If we denote the excentricity by e, we have from the definition,

$$e=\frac{c}{a}, \quad \text{or} \quad c=ae; \quad \therefore \quad b^2=-a^2(1-e^2).$$

Substituting in (7) and (8), we have, for the focal lines of a point whose abscissa is x,

$$r'=a+ex \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$r=-a+ex \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Discussion of the Equation.

65. If we compare equation [43], with equation [31], we see that they are the same, except that b^2 is *negative* in the former and *positive* in the latter. If, therefore, we change $+b^2$ to $-b^2$, in any analytical expression relating to the ellipse, we shall obtain the corresponding analytical expression relating to the hyperbola. But, changing b^2 to $-b^2$, is the same as substituting $b\sqrt{-1}$, for b; hence, when the first power of b enters any expression for the ellipse, we replace it by $b\sqrt{-1}$ to obtain the corresponding expression for the hyperbola.

Proceeding as in Article **47**, and making the changes above indicated, we can show:

1°. The curve cuts the axis of x in the points,

$$(-a,\ 0),\quad (+a,\ 0),$$

but does not cut the axis of y. The last fact is shown, by the ordinates corresponding to the abscissa 0, being imaginary.

2°. Every value of x, between the limits $-a$ and $+a$, gives two imaginary values of y; and every value of x, less than $-a$, or greater than $+a$, gives two real values of y, equal with contrary signs: this shows, that the curve is limited towards the centre by two tangents to the curve, one at each vertex of the transverse axis, and that the curve extends outward from these limits to an infinite distance in both directions, being symmetrical with respect to the axis of x.

For each value of y from $-\infty$ to $+\infty$ there are two real values of x, equal with contrary signs: this shows that the curve is unlimited in the direction of the axis of y, and that it is symmetrical with respect to this axis.

3°. The parameter of the curve, equal to $\frac{4b^2}{2a}$, is a third proportional to the transverse and conjugate axes.

4°. The squares of any two ordinates are to each other as the rectangles of the segments into which they divide the transverse axis.

In this case, the term *segment*, is to be understood in an algebraical sense. Estimating from the left-hand vertex of the axis, the *first* segment is equal to the transverse axis, *plus* the distance from the right-hand vertex

to the foot of the ordinate, and the *second* segment is equal to *minus* the last distance; that is, one segment being *plus* and the other *minus*, their algebraic sum is equal to the transverse axis.

5°. Any point (x', y') is without the hyperbola, upon the hyperbola, or within the hyperbola, according as the expression,

$$a^2y'^2 - b^2x'^2 + a^2b^2,$$

is *greater than, equal to,* or *less than* 0.

A point is without the curve if it lies in the space between the branches.

Conjugate Hyperbolas.

66. Two hyperbolas are conjugate when the transverse axis of one, is the conjugate axis of the other, and the reverse. Thus, the hyperbola whose transverse axis is DD′ is the conjugate of the hyperbola whose transverse axis is AB, and *conversely*, the latter is the conjugate of the former.

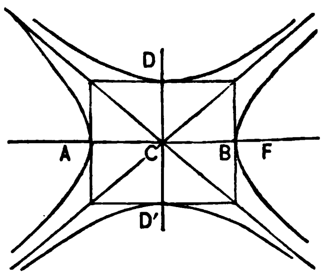

FIG. 44.

The parallelogram described on the axes is common to both curves, and the *four* foci of the two hyperbolas are all on a circle described about C, with the radius CF. To find the equation of the second hyperbola from that

of the first, it is to be observed that the second holds the same relation to the axis of y, that the first does to the axis of x, and the reverse. Hence, if we change a into b, and b into a, and also change x into y, and y into x, in the equation of the first, we shall have the equation of the second. Making these changes in [43], we have,

$$b^2x^2 - a^2y^2 = -a^2b^2,$$

or, by reduction,

$$a^2y^2 - b^2x^2 = a^2b^2 \quad . \quad . \quad . \quad . \quad . \quad [45]$$

Equations [43] and [45] are the equations of two conjugate hyperbolas.

Equation [45] can be reduced to the form,

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [46]$$

It may be shown that every straight line through the centre of any hyperbola intersects, either that hyperbola, or its conjugate. For, let

$$y = \tan\theta\, x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

be the equation of a straight line through the centre. Combining this with equation [43], and finding the simultaneous values of x and y, we have,

and,

$$\left.\begin{aligned} x &= \pm \frac{ab}{\sqrt{b^2 - a^2\tan^2\theta}}, \\ y &= \pm \frac{ab\tan\theta}{\sqrt{b^2 - a^2\tan^2\theta}}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

In like manner, combining (1) with [45], we have,

and,
$$\left.\begin{aligned} x &= \pm \frac{ab}{\sqrt{a^2 \tan^2\theta - b^2}}, \\ y &= \pm \frac{ab\tan\theta}{\sqrt{a^2 \tan^2\theta - b^2}}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3)$$

If, $a^2 \tan^2\theta < b^2$, that is, if $\tan^2\theta < \frac{b^2}{a^2}$, the values of x and y in (2) are real, and the line (1) intersects the given hyperbola; if $\tan^2\theta > \frac{b^2}{a^2}$, the values of x and y in (3) are real, and the line (1) intersects the conjugate hyperbola; if $\tan^2\theta = \frac{b^2}{a^2}$, the values of x and y, in both (2) and (3) are infinite.

In the latter case, the line (1) coincides with one of the diagonals of the rectangle described on the axes of the two conjugate hyperbolas. Hence, if the line (1) falls between these diagonals and the axis of x, it intersects the given hyperbola; if it falls between these diagonals and the axis of y, it intersects the conjugate hyperbola; and if it coincides with either of these diagonals, it intersects both, at an infinite distance from the centre.

We see from equations (2) and (3), that the two points of intersection are equidistant from the centre. Hence, every straight line drawn through the centre, and terminating in either curve, is bisected at the centre.

If we make $a = b$, both the conjugate hyperbolas become *equiaxial*, and their equations reduce to,

$$y^2 - x^2 = -a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$y^2 - x^2 = +a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If the axes of an hyperbola are equal, the hyperbola is said to be *equilateral*. This corresponds to the case in which the ellipse becomes a circle.

Proposition 34.—*To show that the diagonals of the parallelogram on the axes are asymptotes to both curves.*

67. An *asymptote* to a curve is a line that continually approaches the curve, and becomes tangent to it at an infinite distance.

The equations of the diagonals CP and CQ may be written,

$$y = \pm \frac{b}{a} x \; . \; . \; . \; (1)$$

Squaring both members, we have,

$$y^2 = \frac{b^2}{a^2} x^2 \; . \; . \; . \; [47]$$

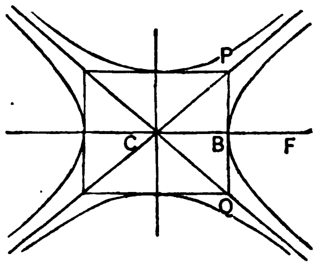

FIG. 45.

If we suppose x to be the same in equations [43], [45], and [47], and denote the values of y in [43] and [45] by y' and y'', to distinguish them from the corresponding value of y in [47], we have, after reduction,

$$y'^2 = \frac{b^2}{a^2} x^2 - b^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$y''^2 = \frac{b^2}{a^2} x^2 + b^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$y^2 = \frac{b^2}{a^2} x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Subtracting (2) from (4), factoring, and reducing, we have,

$$y - y' = \frac{b^2}{y + y'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In like manner, subtracting (4) from (3), and reducing, we have,

$$y'' - y = \frac{b^2}{y'' + y} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The second member of (5) is the distance from any point on the first hyperbola to the diagonal, and the second member of (6) is the corresponding distance from the diagonal to the conjugate hyperbola, these distances being measured on a line parallel to the axis of y. If we suppose y to increase numerically, the values of y' and y'' will also increase numerically, and the corresponding expressions for $y - y'$ and $y'' - y$ will decrease numerically; and finally, when y becomes greater than any assignable value, the expressions for $y - y'$ and $y'' - y$ will become less than any assignable value, or 0; that is, as we proceed outward in either direction from the centre, the two curves approach the diagonals, and finally coincide with, or become tangent to them, at an infinite distance from the centre; hence, the common diagonals are asymptotes to both curves, *which was to be shown.*

Two conjugate hyperbolas are obviously asymptotes to each other. It is also obvious that every diameter—that is, every straight line through the centre, except the diagonals referred to—intersects one of the two conjugate hyperbolas. If its slope is numerically less than $\frac{b}{a}$, it

intersects the first curve; if its slope is numerically greater than $\frac{b}{a}$, it intersects the second curve.

Proposition 35.—*To find the equation of the hyperbola referred to its principal vertex.*

68. In the ellipse, the *left hand vertex* of the transverse axis is the *principal vertex;* in the hyperbola, the *right hand vertex* is the *principal vertex.*

If we make $m = a$ and $n = 0$, in [13] and [14], we have,

$$x = a + x' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$y = y' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting in [43], dropping the dashes from x' and y', and reducing, we have,

$$y^2 = \frac{b^2}{a^2}(2ax + x^2) \quad . \quad . \quad . \quad . \quad . \quad [48]$$

which is the required equation.

This equation might have been found from [33] by changing b^2 to $-b^2$, and $-a$ to $+a$.

Performing the indicated multiplication, substituting for $\frac{2b^2}{a}$ its value $2p$, and denoting the square of the ratio of the semi-axes by r^2, we have,

$$y^2 = 2px + r^2x^2 \quad . \quad . \quad . \quad . \quad . \quad [49]$$

If we make $b = a$, in [48], we have,

$$y^2 = 2ax + x^2.$$

which is the equation of the equilateral hyperbola referred to its principal vertex.

Proposition 36.—*To find the polar equation of the hyperbola when the pole is at either focus.*

69. Assuming equation (12), Article **64**, we have,

$$r = -a + ex \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

From the figure of Article **64**, we have, since CF equals ae,

$$x = ae + r\cos\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting in (1), and solving with respect to r, we have,

$$r = -\frac{a(1-e^2)}{1-e\cos\phi} \quad . \quad . \quad . \quad ; \quad . \quad . \quad [50]$$

which is the polar equation when the pole is at the right-hand focus.

Assuming equation (11), Article **64**, we have,

$$r' = a + ex \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From the figure, we have,

$$x = -ae + r'\cos\phi \quad . \quad . \quad . \quad . \quad (4)$$

Substituting in (3), and solving, we have,

$$r' = \frac{a(1-e^2)}{1-e\cos\phi} \quad . \quad . \quad . \quad . \quad . \quad [51]$$

which is the polar equation of the curve referred to the left-hand focus.

Because e is greater than 1, the values of r and r' in equations (1) and (3) are both positive if x is greater than a, and both negative if x is less than $-a$; that is, they are *both positive* for all points on the *right-hand branch*, and *both negative* for all points on the *left-hand*

branch. Hence, the values of ϕ, that make r and r' positive, in equations [50] and [51], correspond to points on the right-hand branch, and the values that make r and r' negative, correspond to points on the left-hand branch of the curve.

Discussion of the Equations.

70. If we denote the inclination of either asymptote to the transverse axis by θ, we have, from figure 45,

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}} = \frac{1}{e} \quad . \quad \quad . \ . \ . \ (1)$$

Since $e^2 > 1$, the value of r, in equation [50], will be *positive*, that is, it will give points of the right-hand branch, if $e\cos\phi < 1$, or if $\cos\phi < \cos\theta$; r will be negative, that is, it will give points of the left-hand branch, if $e\cos\phi > 1$, or if $\cos\phi > \cos\theta$; and r will be *infinite* if $e\cos\phi = 1$, or if $\cos\phi = \cos\theta$. In like manner, r' will be *positive* if $\cos\phi > \cos\theta$; *negative* if $\cos\phi < \cos\theta$, and *infinite*, if $\cos\phi = \cos\theta$.

If we make $\phi = 0$, we have, after reduction,

$$r = -a(1+e), \quad \text{and} \quad r' = a(1+e).$$

The former determines the left-hand vertex, with respect to the right-hand focus; and the latter determines the right-hand vertex, with respect to the left-hand focus.

In like manner, if we make $\phi = 180°$, and reduce, we have,

$$r = -a(1-e), \quad \text{and} \quad r' = a(1-e),$$

which admit of a corresponding interpretation.

Proposition 37.—*To find the equations of a tangent and normal to the hyperbola.*

71. If we change b^2 to $-b^2$, in equations (7), (8), (9), and (10), of Article **53**, we have,

$$a^2yy' - b^2xx' = -a^2b^2 \quad . \quad . \quad . \quad . \quad . \quad [52]$$

$$y - y' = -\frac{a^2y'}{b^2x'}(x - x') \quad . \quad . \quad . \quad . \quad [53]$$

$$\text{S.T} = \frac{a^2y'^2}{b^2x'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\text{S.N} = \frac{b^2x'}{a^2} \quad . \quad . \quad . \quad . \quad \quad . \quad . \quad (4)$$

Equation [52] is the equation of the tangent; equation [53] is the equation of the normal, and (3) and (4) are expressions for the subtangent and subnormal, corresponding to the point (x', y') of the curve, (Art. **65**).

Since b^2 does not enter equation (11), Article **53**, that equation will be the same for both the ellipse and the hyperbola. Hence, we have for CT, or the abscissa of the point at which the tangent cuts the axis of x,

$$x = \frac{a^2}{x'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This value of x has the same sign as x'; hence, for the right-hand branch it is always positive; that is, the tangent to that branch, cuts the axis to the right of the centre. If x' increases, the value of CT diminishes; and if $x' = \infty$, $\text{CT} = 0$; that is, the tangent to the curve at the point whose abscissa is ∞, passes through the centre, as it should, inasmuch as the tangent to the curve at an infinite distance is an asymptote.

Supplementary Chords and Conjugate Diameters.

72. Two chords are *supplementary*, when they are drawn from the vertices of any diameter to the same point of the curve.

Two diameters are conjugate when either is parallel to a tangent at the vertex of the other.

To find the equations of condition for supplementary chords drawn from the vertices of the transverse axis of the hyperbola, we change b^2 into $-b^2$ in equation [38], Article **54**; this gives,

$$\tan\theta''\tan\theta''' = \frac{b^2}{a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

To find the equation of condition for conjugate diameters, we change b^2 into $-b^2$, in equation [39], Article **54**; this gives,

$$\tan\theta\tan\theta' = \frac{b^2}{a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If $\tan\theta$, is numerically less than $\frac{b}{a}$, either positive or negative, $\tan\theta'$ is numerically greater than $\frac{b}{a}$; that is, if $\tan^2\theta < \frac{b^2}{a^2}$, we shall have $\tan^2\theta' > \frac{b^2}{a^2}$, and consequently from Article **66**, we infer that *if one of two conjugate diameters intersect the given hyperbola, the other intersects its conjugate hyperbola.*

If $\tan\theta = 0$, we have $\tan\theta' = \infty$, which shows that *the axes are conjugate diameters.*

Since the product of $\tan\theta$ and $\tan\theta'$ is positive, they

must have the same signs; that is, both θ and θ' must be acute, or both must be obtuse; hence, *the angle between any two conjugate diameters, other than the axes, is acute.*

As $\tan\theta$ increases numerically, $\tan\theta'$ diminishes numerically; hence, *as one diameter recedes from the transverse axis, the other approaches it.*

If $\tan\theta$ becomes equal to $\pm\frac{b}{a}$, $\tan\theta'$ also becomes equal to $\pm\frac{b}{a}$; hence, *if a diameter coincide with either asymptote, its conjugate coincides with the same asymptote.*

The second members of equations (1) and (2) are equal; placing their first members equal, we have,

$$\tan\theta''\tan\theta''' = \tan\theta\tan\theta' \quad . \quad . \quad . \quad . \quad (3)$$

If $\tan\theta = \tan\theta''$, we have, $\tan\theta' = \tan\theta'''$, and the reverse. Hence, if a diameter is parallel to a chord, the conjugate of that diameter is parallel to the supplement of the chord, and the reverse.

This principle gives rise to the following constructions:

1°. *To draw a tangent to an hyperbola at a given point.*—Let P be the given point; draw CP, and through A draw a chord parallel to CP, and prolong it till it intersects the curve at H; then draw the chord HB supplementary to AH; through P draw PT parallel to HB; this line is obviously, the required tangent.

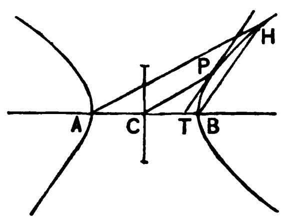

FIG. 46.

2°. *To draw a tangent to a hyperbola parallel to a given line.*—Draw the chord BH parallel to the given line; draw its supplement, AH; then draw a semi-diameter, CP, parallel to AH, meeting the curve at P; through P draw PT parallel to HB, and it will be the tangent required.

Proposition 38.—*To prove that a tangent to the hyperbola bisects the angle between the focal lines to the point of contact.*

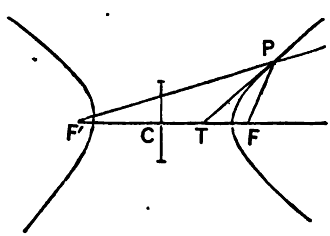

FIG. 47.

73. Denoting the co-ordinates of P, by x' and y', we have for CT, (Art. **71**),

$$x = \frac{a^2}{x'} \dots (1)$$

Adding this to, and subtracting it from, F′C, or ae, we have, after reduction,

$$\text{F'T} = \frac{a}{x'}(ex' + a) \quad \text{and} \quad \text{FT} = \frac{a}{x'}(ex' - a) \ . \ . \ (2)$$

But from equations (11) and (12), Article **64**, we have, for the focal lines to the point (x', y'),

$$r' = ex' + a, \quad \text{and} \quad r = ex' - a \ . \ . \ . \ (3)$$

Hence, F′T and FT are equimultiples of r' and r; consequently, we have,

$$r' : r :: \text{F'T} : \text{FT} \ . \ . \ . \ . \ . \ (4)$$

The tangent therefore divides the base of the triangle PF′F into segments proportional to the adjacent sides;

hence, it bisects the angle between those sides, *which was to be proved.*

This principle gives rise to the following constructions:

1°. *To draw a tangent to an hyperbola at a given point.*—Let P be the given point. Draw the focal lines PF and PF′, and draw PT bisecting the angle between them; this line is the required tangent.

2°. *To draw a tangent to an hyperbola through a point without the curve.*—Let H be the point. With H as a centre on a radius HF, describe an arc; then, with F′ as a centre, and with the transverse axis as a radius, describe a second arc, cutting the first in G and G′; draw F′G, and prolong it till it meets the curve at P; draw HPT, and it will be the required tangent.

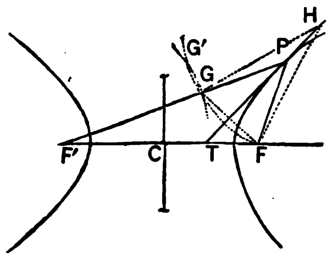

FIG. 48.

For, $HF = HG$ by construction; and since P is a point of the curve and $F'G = 2a$, we have $PG = PF$; hence, PT has two of its points, P and H, equally distant from F and G; it is therefore perpendicular to FG, and consequently bisects the angle F′PF; hence, it is the required tangent.

A second point of contact might be found by drawing a line from F′ to G′, and prolonging it to meet the curve.

Proposition 39.—*To find the equation of an hyperbola referred to its centre and any pair of conjugate diameters.*

74. Assume the equation of the hyperbola referred to its centre and axes,

$$a^2y^2 - b^2x^2 = -a^2b^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The formulas for changing the directions of the coordinate axes without changing the origin, are,

$$x = x' \cos a + y' \cos a' \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$y = x' \sin a + y' \sin a' \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Substituting these values of x and y in (1), we have, after arranging and factoring,

$$\begin{aligned}(a^2 \sin^2 a' - b^2 \cos^2 a')\, y'^2 + (a^2 \sin^2 a - b^2 \cos^2 a)\, x'^2 \\ + 2\,(a^2 \sin a \sin a' - b^2 \cos a \cos a')\, x'y' = -a^2b^2 \, . \, . \, (4)\end{aligned}$$

which is the equation of the hyperbola referred to any oblique axes passing through the centre.

If the new axes are made conjugate diameters we must have, (Art. **72**), changing θ and θ' to a and a',

$$\tan a \tan a' = \frac{b^2}{a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

or,

$$\frac{\sin a \ \sin a'}{\cos a \ \cos a'} = \frac{b^2}{a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Clearing of fractions and transposing, we have,

$$a^2 \sin a \sin a' - b^2 \cos a \cos a' = 0 \quad . \quad . \quad . \quad (7)$$

Substituting in (4), we have,

$$(a^2 \sin^2 a' - b^2 \cos^2 a')y'^2 + (a^2 \sin^2 a - b^2 \cos^2 a)x'^2 = -a^2b^2 \, . \, (8)$$

which is the equation of the hyperbola referred to any pair of conjugate diameters.

To reduce this equation to the same form as equation (1), let us denote the semi-conjugate diameters by a' and b', and furthermore let us suppose that the first one meets the curve, in which case we know that the second one does not meet it, but terminates in the conjugate hyperbola, (Art. **72**).

If we make $y' = 0$, in equation (8), the corresponding value of x'^2 will be equal to a'^2; if we make $x' = 0$, the corresponding value of y'^2 will be equal to $-b'^2$.

Making these suppositions, we have,

$$a'^2 = -\frac{a^2b^2}{a^2\sin^2 a - b^2\cos^2 a} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$-b'^2 = -\frac{a^2b^2}{a^2\sin^2 a' - b^2\cos^2 a'} \quad . \quad . \quad . \quad (10)$$

From (9) and (10), we find,

$$a^2\sin^2 a - b^2\cos^2 a = -\frac{a^2b^2}{a'^2} \quad . \quad . \quad . \quad (11)$$

and,

$$a^2\sin^2 a' - b^2\cos^2 a' = \frac{a^2b^2}{b'^2} \quad . \quad . \quad . \quad . \quad (12)$$

Substituting in (8), dropping the dashes from x and y, and reducing, we have,

$$a'^2y^2 - b'^2x^2 = -a'^2b'^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

If we had transformed the equation of the conjugate hyperbola, we should have an equation the same as equation (8), except the sign of the second member,

which would have been positive. From that we could find, in like manner, the equation of the conjugate hyperbola referred to the same conjugate diameters,

$$a'^2y^2 - b'^2x^2 = a'^2b'^2 \quad . \quad . \quad . \quad . \quad . \quad [54]'$$

These equations are of the same form as the corresponding ones referred to the centre and axes. Comparing [54] with [41], the equation of the ellipse referred to conjugate diameters equation, (Art. **57**), we see that they differ from each other only in the sign of b'^2. Hence, any analytical property of the hyperbola may be deduced from the corresponding property of the ellipse by simply changing $+b'^2$, to $-b'^2$, or, which is the same thing, by changing b', to $b'\sqrt{-1}$. Thus, if we change b^2, to $-b^2$, and b'^2, to $-b'^2$, in equation (11), Article **59**, we have,

$$a'^2 - b'^2 = a^2 - b^2.$$

That is, *the difference of the squares of any two semi-conjugate diameters of an hyperbola is equal to the difference of the squares of the semi-axes.*

Proposition 40.—*To find the equation of a tangent to an hyperbola referred to conjugate diameters.*

75. If we change b'^2 into $-b'^2$, in equation [42], (Art. **60**), we have,

$$a'^2yy' - b'^2xx' = -a'^2b'^2 \quad . \quad . \quad . \quad . \quad (1)$$

which is the equation of a tangent to the hyperbola referred to conjugate diameters, as axes.

To find the point in which this tangent cuts the axis of x, we make $y = 0$ in (1), which gives,

$$x = \frac{a'^2}{x'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This value of x is equal to 0, if x' is infinite, and equal to a', if x' equals a'.

Of Poles and Polars.

76. It may be shown by a course of reasoning entirely similar to that used in Article **61**, that if any pair of conjugate diameters be assumed as axes of co-ordinates, and any line be drawn parallel to the one taken as the axis of y, and if a pair of tangents be drawn to the curve from any point of this line, the corresponding chords of contact will all intersect at a common point of the axis of x. This point is called the *pole* of the line, and the line is called the *polar* of the point.

Every point in the plane of the curve has a *polar*, and every straight line in that plane has a *pole*. The methods of constructing the *pole of a line*, and the *polar of a point*, are entirely analogous to those given for the corresponding cases in Article **61**.

As in the ellipse, the *polar* of either focus is a directrix of the curve, that is, it is a line such that the distance of any point of the curve from it bears a *constant ratio* to the corresponding focal distance of the same point.

The constant ratio is the *excentricity* of the hyperbola.

Proposition 41.—*To find the equation of the hyperbola, when referred to its asymptotes.*

77. Let us take the asymptote CQ as the axis of x, and the asymptote CP as the axis of y. Then will

$$\sin a = -\frac{BQ}{CQ} = -\frac{b}{\sqrt{a^2 + b^2}} \quad . \quad . \quad . \quad . \quad (1)$$

$$\cos a = \frac{CB}{CQ} = \frac{a}{\sqrt{a^2 + b^2}} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\sin a' = \frac{BP}{CP} = \frac{b}{\sqrt{a^2 + b^2}} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\cos a' = \frac{CB}{CP} = \frac{a}{\sqrt{a^2 + b^2}} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From these equations we find,

$$a^2 \sin^2 a' - b^2 \cos^2 a' = 0;$$

$$a^2 \sin^2 a - b^2 \cos^2 a = 0; \text{ and}$$

$$a^2 \sin a \sin a' - b^2 \cos a \cos a' = -\frac{2a^2b^2}{a^2 + b^2};$$

which are equations of condition for asymptotes.

Substituting these in equation (4), Article **74**, which is the equation of the hyperbola referred to any oblique axes through the centre, we have,

$$-\frac{4a^2b^2}{a^2 + b^2} x'y' = -a^2b^2 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which is the equation of the hyperbola referred to its asymptotes. Dropping the dashes from x' and y', and reducing, we have,

$$xy = \frac{a^2 + b^2}{4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Denoting the value of the second member* by m, we have,

$$xy = m \quad . \quad . \quad [55]$$

which is the required equation.

Making the second member of (5) positive, and proceeding as before, we have,

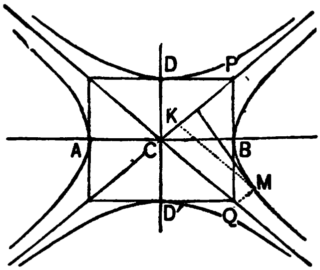

FIG. 49.

$$xy = -m \quad . \quad . \quad . \quad . \quad . \quad . \quad [55]'$$

which is the equation of the conjugate hyperbola referred to the same asymptotes.

Discussion of the Equation.

78. If we solve equation [55] first with respect to x, and then with respect to y, we have,

$$x = \frac{m}{y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$y = \frac{m}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From equation (1) we see that as y increases, x diminishes; and when $y = \infty$, $x = 0$. This shows that the curve approaches the axis of y, and finally becomes tangent to it, at an infinite distance from the centre. From equation (2) we infer, in like manner, that the curve approaches the axis of x, and finally becomes tangent to it, at an infinite distance from the centre.

The second member of equation [55] is *essentially positive;* hence, both x and y have the same sign.

This shows that one branch lies wholly above the axis of x, and to the right of the axis of y, and the other wholly below the axis of x, and to the left of the axis of y.

In like manner, it can be shown from equation [55]′, that one branch of the *conjugate hyperbola* lies wholly above the axis of x, and to the left of the axis of y, and the other wholly below the axis of x, and to the right of the axis of y.

If we suppose $a = b$, the hyperbolas become *equilateral*, and the asymptotes are then perpendicular to each other. In what follows, we shall suppose the hyperbolas to be *equiaxial*, or *equilateral.*

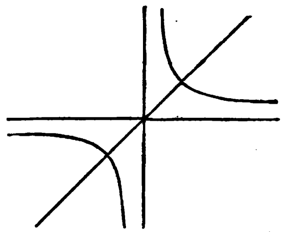

FIG. 50.

Proposition 42. — *To find the equation of a tangent to an equilateral hyperbola, referred to the asymptotes.*

79. Assume the equation of the hyperbola, referred to its asymptotes,

$$xy = m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The conditions that place the points (x', y') and (x'', y'') on the curve, are,

$$x'y' = m \quad \text{and} \quad x''y'' = m.$$

Subtracting the first of these from the second, member from member, we have,

$$x''y'' - x'y' = 0.$$

Adding and subtracting $x''y'$, and factoring, we have,

$$x''(y'' - y') + y'(x'' - x') = 0 \quad . \quad . \quad . \quad (2)$$

whence,

$$\frac{y''-y'}{x''-x'}=-\frac{y'}{x''} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The second member of (3) is the slope of the secant; if we make $y''=y'$ and $x''=x'$, we find for the slope of the tangent,

$$\tan\theta'=-\frac{y'}{x'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Substituting in [23], we have, for the equation of the tangent,

$$y-y'=-\frac{y'}{x'}(x-x') \quad . \quad . \quad . \quad . \quad [56]$$

Substituting in [25], we find for the subtangent,

$$\mathrm{S.T}=-x' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This shows that CQ=QT; and since the triangles BCT and PQT are similar, we have BP = PT. Hence, that portion of a tangent, which is intercepted between the asymptotes is bisected at the point of contact.

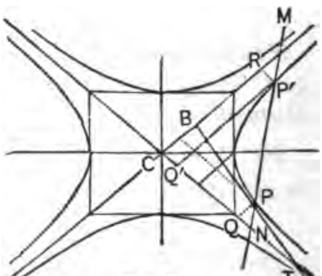

Fig. 51.

Proposition 43.—*To show that the intercepts of a secant, between the curve and its asymptotes are equal.*

80. If in equation [5] we make the coefficient of the second member equal to the second member of equation (3), Article **79**, we have,

$$y - y' = -\frac{y'}{x''}(x - x') \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which is the equation of a secant through the points (x', y') and (x'', y''). Let these points be P and P′, respectively (Fig. 51); then will

$$CQ = x', \quad CQ' = RP' = x'', \quad QP = y', \quad \text{and} \quad Q'P' = y''.$$

If we make $y = 0$ in equation (1), the corresponding value of x will be equal to CN. Making the substitution and reducing, we have,

$$CN = x' + x'' \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence,

$$CN - x' = x'';$$

or,

$$QN = RP'.$$

Now the triangles QNP and RP′M are mutually equiangular; and since their homologous sides, QN and RP′, are equal, the triangles are equal in all their parts; hence, $NP = P'M$, *which was to be proved.*

This principle gives rise to a method of constructing the curve, when its asymptotes and one point are given.

Through the given point, draw any straight line, intercepted by the asymptotes; then lay off on this line, from one asymptote, a distance equal to the given point from the other asymptote; the point thus found will be a point of the curve. Find in this manner any number of points, and draw a line through them; *this is the required line.*

PROBLEMS.

1°. Find the axes of the hyperbola whose equation is,

$$3y^2 - 2x^2 = -12.$$

Ans. $a = \sqrt{6}$, and $b = 2$.

2°. Find the excentricity of the hyperbola,

$$3y^2 - 2x^2 = -12,$$

and also the excentricity of its conjugate.

Ans. $e = \sqrt{\frac{5}{3}}$, and $e' = \sqrt{\frac{5}{2}}$.

3°. Find the intersection of the hyperbola,

$$3y^2 - 2x^2 = -12,$$

and the circle,

$$x^2 + y^2 = 16.$$

Ans. $(\pm 2\sqrt{3}, \pm 2)$.

4°. Find the parameter of the hyperbola,

$$3y^2 - 2x^2 = -12.$$

Ans. $2p = \frac{8}{\sqrt{6}}$.

5°. Determine the position of the point (2, 3) with respect to the hyperbola,

$$4y^2 - 2x^2 = -9.$$

Ans. It lies without the curve.

6°. Find the angle included between the asymptotes of the hyperbola,

$$16y^2 - 9x^2 = -25.$$

Ans. 73° 44′.

7°. Find whether the line $y = \frac{3}{4}x$ intersects the hyperbola $5y^2 - 2x^2 = -15$, or its conjugate.

Ans. It intersects the conjugate.

8°. Find expressions for the subtangent and subnormal of the hyperbola,

$$3y^2 - x^2 = -9,$$

at the point whose abscissa is 6.

Ans. $S.T = 4\frac{1}{2}$, and $S.N = 2$.

9°. Find the perpendicular distance from the focus of any hyperbola to its asymptote.

Ans. The semi-conjugate axis.

10°. Find the equation of a tangent to an hyperbola in terms of its slope and the semi-axes.

Solution.—From equation (2), problem 10°, on the ellipse, (Art. **61**), we have, by changing b^2 to $-b^2$,

$$y = x \tan\theta + \sqrt{a^2 \tan^2\theta - b^2} \quad . \quad . \quad . \quad . \quad (1)$$

which is the required equation.

11°. To find the *locus* of the intersection of any tangent and a perpendicular to it, from the focus.

Solution.—By a process entirely analogous to that employed in the solution of Problem 11°, on the ellipse, (Art. **61**), we find,

$$x^2 + y^2 = a^2.$$

Hence, the required locus is a circle described on the transverse axis as a diameter.

12°. Find the position of the point (x', y'), so that its distance from any point of the hyperbola shall be rational in terms of the abscissa of that point.

Ans. It must be at one of the foci.

IX. LINES OF THE SECOND ORDER.

Classification of Lines.

81. Lines lying in a single plane are classed according to the degree of their equations.

A line of *the first order*, is one whose equation is of the first degree.

It has been shown that *every line of the first order is a straight line.*

A line of *the second order*, is one whose equation is of the second degree.

It has been shown that the *circle*, the *parabola*, the *ellipse*, and the *hyperbola*, are lines of the second order; it remains to be shown that *every line of the second order is one of these curves.*

General Equation of the Second Degree.

82. Every equation of the second degree between two variables, is a particular case of the general equation,

$$ay^2 + bxy + cx^2 + dy + ex + f = 0 \quad . \quad . \quad (1)$$

in which a, b, c, &c., are *arbitrary constants.*

We shall suppose the axes of co-ordinates rectangular;

for, if they were oblique, we might transform the equation to one in which the axes were rectangular without affecting its *form*, or *degree*.

In order to determine the particular lines represented by equation (1), under the various hypotheses that may be made on the constants that enter it, we shall subject it to a series of transformations, the object of which is to reduce it to some known form.

First Transformation.

83. The object of this transformation is to get rid of the term containing the product of x and y. To accomplish this, we refer the line to a new set of rectangular axes, having the same origin as before. The formulas for making this transformation are found by making m and n equal to 0, in equations [11] and [12]. This change gives,

$$x = x'\cos a - y'\sin a \quad . \quad . \quad . \quad . \quad . \quad [11]$$

$$y = x'\sin a + y'\cos a \quad . \quad . \quad . \quad . \quad . \quad [12]$$

Substituting these values of x and y in equation (1), and arranging the terms of the resulting equation, we have,

$$\left.\begin{array}{l} a\cos^2 a \\ +\, c\sin^2 a \\ -\, b\sin a\cos a \end{array}\right| y'^2 + \left.\begin{array}{l} 2(a-c)\sin a\cos a \\ \quad +\, b(\cos^2 a - \sin^2 a) \\ \end{array}\right| x'y' + \left.\begin{array}{l} a\sin^2 a \\ +\, c\cos^2 a \\ +\, b\sin a\cos a \end{array}\right| x'^2$$

$$+ \left.\begin{array}{l} d\cos a \\ -\, e\sin a \end{array}\right| y' + \left.\begin{array}{l} d\sin a \\ +\, e\cos a \end{array}\right| x' + f = 0 \quad . \quad . \quad . \quad (2)$$

Since a is arbitrary, we may assign to it such a value

as will reduce the coefficient of $x'y'$ to 0; that is, we may assume,

$$2(a-c)\sin a\cos a + b(\cos^2 a - \sin^2 a) = 0.$$

But, we know from trigonometry that,

$$2\sin a\cos a = \sin 2a, \quad \text{and} \quad \cos^2 a - \sin^2 a = \cos 2a.$$

Substituting these in the preceding equation, and transposing, we have,

$$(a-c)\sin 2a = -b\cos 2a.$$

Whence,

$$\tan 2a = \frac{b}{c-a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From this value of $\tan 2a$ the value of a may be found, and since $\tan 2a$ is real, the first transformation is always possible.

Denoting the corresponding values of the coefficients of the different terms of equation (2), by a', c', d', &c., and dropping the dashes from the variables, we have, for the *first transformed equation,*

$$a'y^2 + c'x^2 + d'y + e'x + f = 0 \quad . \quad . \quad . \quad (4)$$

To find the values of a' and c', in equation (4), we have,

$$a' = a\cos^2 a + c\sin^2 a - b\sin a\cos a;$$

$$c' = a\sin^2 a + c\cos^2 a + b\sin a\cos a.$$

But,

$$a\cos^2 a = \frac{1}{2}a\cos^2 a + \frac{1}{2}a(1-\sin^2 a) = \frac{1}{2}a + \frac{1}{2}a(\cos^2 a - \sin^2 a);$$

$$c\sin^2 a = \frac{1}{2}c\sin^2 a + \frac{1}{2}c(1-\cos^2 a) = \frac{1}{2}c - \frac{1}{2}c(\cos^2 a - \sin^2 a).$$

Hence, by substitution and reduction,

$$a' = \frac{1}{2}[c + a - (c - a)\cos 2a - b\sin 2a.]$$

And in like manner,

$$c' = \frac{1}{2}[c + a + (c - a)\cos 2a + b\sin 2a].$$

Now we know from trigonometry that,

$$\cos 2a = \frac{1}{\sqrt{1 + \tan^2 2a}} = \frac{c - a}{\sqrt{(c - a)^2 + b^2}}$$

$$\sin 2a = \frac{\tan 2a}{\sqrt{1 + \tan^2 2a}} = \frac{b}{\sqrt{(c - a)^2 + b^2}}$$

Substituting these in the values of a' and c', we have,

$$a' = \frac{1}{2}[c + a \mp \sqrt{(c - a)^2 + b^2}] \quad . \quad . \quad . \quad (5)$$

$$c' = \frac{1}{2}[c + a \pm \sqrt{(c - a)^2 + b^2}] \quad . \quad . \quad . \quad (6)$$

Second Transformation.

84. The object of this transformation is to get rid of the terms containing the first powers of x and y. To accomplish this, we refer the line to a set of rectangular axes, parallel to the first set, but having a different origin. The formulas for making this transformation are,

$$x = m + x' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [13]$$

$$y = n + y' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [14]$$

Substituting these values for x and y, in equation (4), and arranging the terms of the resulting equation, we have,

$$a'y'^2+c'x'^2+\begin{array}{l|l|l|}2a'n & y'+2c'm & x'+a'n^2 \\ +d' & +e' & +c'm^2 \\ & & +d'n \\ & & +e'm \\ & & +f\end{array}=0 \quad . . . (7)$$

Since m and n are entirely arbitrary, we may, in general, attribute such values to them as to make the coefficients of x' and y' equal to 0, or, to satisfy the equations,

$$2a'n+d'=0 \quad \text{and} \quad 2c'm+e'=0.$$

That is, we may make,

$$n=-\frac{d'}{2a'}, \quad \text{and} \quad m=-\frac{e'}{2c'} \quad . \quad . \quad . \quad (8)$$

When neither a' nor c' is equal to 0, the transformation *can* be made. In this case, denoting the absolute term by $-f'$, and dropping the dashes from x' and y', we have, for the *second transformed equation,*

$$a'y^2+c'x^2=f' \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

When either a' or c' is equal to 0, the new origin will be at an infinite distance, and the second transformation becomes impossible. We cannot, however, suppose that both a' and c' are equal to 0 at the same time, because that would reduce equation (7) to an equation of the *first degree.* Let us, then, suppose that one of them, as c', is equal to 0. In this case, the first transformed equation reduces to,

$$a'y^2+d'y+e'x+f=0 \quad . \quad . \quad . \quad . \quad (10)$$

Let this equation be transformed so as to get rid of the second and fourth terms. For this purpose, substitute for x and y their values taken from equations [13] and [14]. Substituting and arranging the terms of the resulting equation, we have,

$$a'y'^2 + \left.\begin{array}{l} 2a'n \\ + d' \end{array}\right| y' + e'x' + \left.\begin{array}{l} a'n^2 \\ + d'n \\ + e'm \\ + f \end{array}\right| = 0 \quad . \quad . \quad (11)$$

Making the coefficient of y' and the absolute term each equal to 0, we have,

$$2a'n + d' = 0, \quad \text{and} \quad a'n^2 + d'n + e'm + f = 0;$$

whence, by combination and reduction,

$$n = -\frac{d'}{2a'}, \quad \text{and} \quad m = \frac{d'^2 - 4a'f}{4a'e'} \quad . \quad . \quad (12)$$

These values of n and m are finite, and the transformation can be made, except when $e' = 0$. *First*, suppose e' is not equal to 0. In this case, dropping the dashes from x and y in (11), substituting for m and n their values taken from (12), transposing, and reducing, we have,

$$y^2 = -\frac{e'}{a'}x \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Secondly, suppose $e' = 0$. In this case, equation (10) reduces to,

$$a'y^2 + d'y + f = 0;$$

whence, by solution,

$$y = -\frac{d'}{2a'} \pm \sqrt{-\frac{f}{a'} + \frac{d'^2}{4a'^2}};$$

or,

$$y = \frac{1}{2a'}\{-d' \pm \sqrt{d'^2 - 4a'f}\} \quad . \quad . \quad . \quad (14)$$

Discussion.

85. It has been shown in the preceding articles, that the general equation, (1), of the second degree between two variables, may be reduced to one of the three forms shown in equations (9), (13), and (14).

In the discussion of these equations, there may be three cases:

1°. We may have $b^2 = 4ac$.

In this case, the quantity under the radical sign, in equations (5), and (6), becomes equal to the square of $c + a$, and either a' or c' reduces to 0; if we suppose $c' = 0$, as in Art. **84**, we shall have $a' = a + c$.

2°. We may have, $b^2 < 4ac$.

In this case, the radical in (5), and (6), is less than $c + a$; consequently, a' and c' have the same sign.

3°. We may have $b^2 > 4ac$.

In this case, the radical is greater than $c + a$; consequently, a' and c' have contrary signs.

First Case: $b^2 = 4ac$.

86. In this case, the general equation may be reduced either to the form,

$$y^2 = -\frac{e'}{a'}x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

or, to the form,

$$y = \frac{1}{2a'}\{-d' \pm \sqrt{d'^2 - 4a'f}\} \quad . \quad . \quad . \quad (14)$$

Equation (13) is the equation of a parabola whose parameter is $-\frac{e'}{a'}$. This parabola lies to the right of the axis of y, if e' and a' have contrary signs, and to the left of that axis, if e' and a' have the same signs.

Equation (14) is independent of x, that is, it is true, whatever may be the value of x; hence, it is the equation of two straight lines parallel to the axis of x. These lines become *coincident*, if $d'^2 = 4a'f$, and they are said to be *imaginary*, if $d'^2 < 4a'f$.

Two parallel straight lines, one straight line, and *two imaginary straight lines*, are regarded as *limiting cases* of the parabola.

Second Case; $b^2 < 4ac$.

87. In this case, both a' and c' have the same sign, and equation (9), has the form,

$$a'y^2 + c'x^2 = f' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

We may regard both a' and c' as positive; for, if they were not, they might be made so, by dividing both members of the equation by -1.

Dividing both members of equation (9) by f', we have,

$$\frac{a'}{f'}y^2 + \frac{c'}{f'}x^2 = 1 \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Comparing this with equation [32], we see that it has the same form; hence, *equation* (15) *is the equation of an ellipse.*

If we denote the semi-axes of this ellipse by A, and B, we have,

$$A = \sqrt{\frac{f'}{c'}}, \quad \text{and} \quad B = \sqrt{\frac{f'}{a'}}.$$

If a' and c' are equal, the values of A and B are also equal, and the curve is *a circle.*

If f' is essentially negative, the values of A and B are imaginary, and the corresponding curve, is called *an imaginary curve.*

If f' is equal to 0, the values of A and B are also equal to 0, and the curve reduces to *a point.*

The *circle,* the *point,* and the *imaginary curve,* are regarded as *limiting cases* of the ellipse.

Third Case; $b^2 > 4ac$.

88. In this case a' and c' have contrary signs, and equation (9), takes the form,

$$a'y^2 - c'x^2 = f' \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

We may regard a' as positive; for, were it not, it might be made so, by dividing both members of the equation by -1.

Dividing both members of (16) by f', we have,

$$\frac{a'}{f'}y^2 - \frac{c'}{f'}x^2 = 1 \quad . \quad . \quad . \quad . \quad . \quad (17)$$

In this equation f' may be *essentially negative;* or it may be *essentially positive;* or it may be *equal to* 0.

First, suppose f' *essentially negative;* giving to it its

proper sign, and then dividing both members by -1, we have,

$$\frac{a'}{f'}y^2 - \frac{c'}{f'}x^2 = -1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

which is of the same form as [44]. Hence, equation (18), is the equation of an hyperbola.

Secondly, suppose f' *essentially positive;* in this case, we have,

$$\frac{a'}{f'}y^2 - \frac{c'}{f'}x^2 = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

which is of the same form as [46]. Hence, equation (19) is the equation of an hyperbola, conjugate with the hyperbola (18).

If we denote the semi-axes of these curves by A and B, we have,

$$\mathrm{A} = \sqrt{\frac{f'}{c'}}, \quad \text{and} \quad \mathrm{B} = \sqrt{\frac{f'}{a'}}.$$

If a' equals c', the values of A and B are equal, and the hyperbola is *equilateral.*

Thirdly, suppose f' to be *equal to* 0; in this case, equation (16), reduces to the form,

$$y^2 = \frac{c'}{a'}x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Whence,

$$y = \pm\sqrt{\frac{c'}{a'}}x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

which is the equation of two straight lines that intersect each other at the origin.

The *equilateral hyperbola*, and *two intersecting straight lines*, are regarded as *limiting cases* of the hyperbola.

From what precedes, we conclude that the general equation of the second degree, between two variables, always represents, a *parabola*, an *ellipse*, an *hyperbola*, or some one of their limiting cases.

If $b^2 = 4ac$, it represents a *parabola*; if $b^2 < 4ac$, it represents an *ellipse*; if $b^2 > 4ac$, it represents an *hyperbola*.

We may readily deduce a general expression for the excentricity of a line of the second order. From the definition of the excentricity of the hyperbola, and ellipse, we have,

$$e = \sqrt{\frac{A^2 \pm B^2}{A^2}},$$

in which A and B are the semi-axes. Substituting for A^2 and B^2, their values as found above, and reducing, we have,

$$e = \sqrt{\frac{a' \pm c'}{a'}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The upper sign corresponds to the hyperbola, the lower sign to the ellipse, and making $c' = 0$, we have the excentricity of the parabola.

In the ellipse, the excentricity may have any value, from 0 to 1; in the parabola, it is always equal to 1; and in the hyperbola, it may have any value from 1 to ∞.

Of Centres.

89. A *centre* of a curve is a point that bisects every straight line passing through it, and terminating in the curve.

If the origin of co-ordinates be taken at the centre, the equation of the curve must be of such a form that it will not be changed by substituting $-x$, and $-y$, for $+x$, and $+y$; that is, it must be *homogeneous* with respect to the variables that enter it.

Let us resume equation (1), and transform it so as to get rid of the terms that are of the first degree with respect to x and y. The formulas for making this transformation are,

$$x = m + x' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [13]$$

$$y = n + y' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [14]$$

Substituting these values of x and y, in equation (1), and arranging the resulting equation, we have,

$$\begin{array}{rl|rl|rl|l} ay'^2 + bx'y' + cx'^2 & +2na & y' & +2mc & x' & +an^2 & = 0 \quad . \quad . \quad (22) \\ & +d & & +e & & +bmn & \\ & +bm & & +bn & & +cm^2 & \\ & & & & & +dn & \\ & & & & & +em & \\ & & & & & +f & \end{array}$$

Putting the coefficients of x' and y' equal to 0, we have,

$$2an + bm + d = 0,$$

$$bn + 2cm + e = 0.$$

Combining these equations, we find,

$$m = \frac{2ae - bd}{b^2 - 4ac} \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$n = \frac{2cd - be}{b^2 - 4ac} \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Making these substitutions in equation (22), representing the absolute term by, $-f'$, and dropping the dashes from the variables, we have,

$$ay^2 + bxy + cx^2 = f' \quad . \quad . \quad . \quad . \quad . \quad (25)$$

If we change x, and y, to $-x$, and $-y$, the form of (25), will not be changed. Hence, the origin of co-ordinates is the centre of the curve, and the values of m and n, in (23) and (24), are its co-ordinates when referred to the primitive system.

The values of m and n, are finite for the ellipse and hyperbola, but they are both infinite for the parabola. Hence, each of the former curves has a centre, but the latter has no centre, at a finite distance.

PROBLEMS.

1°. Determine the species of the curve whose equation is,

$$5y^2 + 4xy + 3x^2 - 7y - 2x - 4 = 0.$$

Solution.—Here, $b = 4$, $a = 5$, and $c = 3$; hence,

$$b^2 < 4ac.$$

The curve is therefore an ellipse.

2°. Find the position of its centre, and the inclination of its principal axis.

Solution.—We have, $a = 5$, $b = 4$, $c = 3$, $d = -7$, and $e = -2$. Hence, from (23) and (24), we have,

$$m = -\frac{2}{11},\ n = +\frac{17}{22};$$

that is, *the centre is at the point* $\left(-\frac{2}{11}, \frac{17}{22}\right)$

To find the slope of the principal axis, we have, from equation (3),

$$\tan 2a = -\frac{4}{2}; \quad \therefore \quad 2a = 116° \ 34', \quad \text{or,} \quad a = 58° \ 17';$$

that is, *the inclination of the principal axis is* 58° 17′, *with respect to the primitive axis of abscissas.*

3°. Determine the species of the curve,

$$y^2 - 2xy + 2x^2 - 2y + 2x = 0,$$

and find the points in which it intersects the axis of x.

Ans. The curve is an ellipse. It intersects the axis of x in the points (0, 0) and (— 1, 0).

4°. Find the points in which the curve,

$$y^2 - 2xy + 2x^2 - 2x = 0,$$

cuts the co-ordinate axes.

Ans. It cuts the axis of x in the points (0, 0) *and* (1, 0), *and is tangent to the axis of y at the origin.*

Determine the species of the following curves:

5°. $y^2 - xy - 2x^2 - y - 3x = 0.$

Ans. It is an hyperbola.

6°. $y^2 - 2xy + x^2 - x - y - 1 = 0.$

Ans. A parabola.

7°. $y^2 - 2xy + 2x^2 + 2y + x + 3 = 0.$

Ans. An ellipse.

8°. Find the equation of the diameter of the curve,

$$y^2 - 2xy + 2x^2 - 2x = 0 \quad . \quad . \quad . \quad . \quad (1)$$

that bisects all the chords parallel to the axis of y.

Solution.—If we solve equation (1), with respect to y, we have,

$$y = x \pm \sqrt{-x(x-2)} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For every value of x between 0 and 2, there are two corresponding values of y, one equal to that value of x, *plus* the corresponding value of the radical, and the other equal to that value of x, *minus* the corresponding value of the radical; that is, the line whose equation is,

$$y = x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

is such, that if from any point whose abscissa is greater than 0 and less than 2, we lay off a distance upward, equal to the corresponding value of the radical, and also a distance downward, equal to the corresponding value of the radical, we shall in both cases, determine points of the curve. Hence, the line (3), is the equation of a diameter, *which was to be found.*

9°. Find the length of this diameter, and also the length of its conjugate.

Solution.—Combining equation (3), with the equation of the curve, we find for the two points of intersection,

$$(0, 0), \quad \text{and} \quad (2, 2).$$

The distance between these points is equal to $2\sqrt{2}$; this is the length of the first diameter.

To find the length of the second diameter, make x equal to 1, the abscissa of the centre; the corresponding values of y are,

$$y = 0, \quad \text{and} \quad y = 2.$$

Hence, the extremities of the diameter parallel to the axis of y, are,

$$(1, 0), \quad \text{and} \quad (1, 2),$$

and the distance between these points, that is, the length of the corresponding diameter, is equal to 2.

10°. Determine the species of curve of the second order, that passes through the points $(0, 1)$ and $(0, 2)$ of the axis of y, and through the points $(2, 0)$ and $(3, 0)$ of the axis of x.

Solution.—Dividing both members of equation (1) by f, and denoting the resulting constants by a', b', c', &c., we have,

$$a'y^2 + b'xy + c'x^2 + d'y + e'x + 1 = 0 \quad . \quad (1)$$

Making, $x = 0$, we have,

$$a'y^2 + d'y = -1 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Making, $y = 0$, we have,

$$c'x^2 + e'x = -1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The roots of equation (2) must be equal to 1, and 2, and the roots of (3) must be equal to 2 and 3. Hence, we have the following equations of condition:

$$a' + d' = -1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$4a' + 2d' = -1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$4c' + 2e' = -1 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$9c' + 3e' = -1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Combining and solving, we find,

$$a' = \frac{1}{2}, \quad d' = -\frac{3}{2}, \quad c' = \frac{1}{6}, \quad \text{and} \quad e' = -\frac{5}{6}.$$

Substituting in (1), multiplying by 6, and reducing, we have,

$$3y^2 + 6bxy + x^2 - 9y - 5x + 6 = 0 \quad . \quad . \quad (8)$$

which is the equation of the curve; and since b is entirely arbitrary, we may give to it such a value as will make the curve, (8), either an ellipse, an hyperbola, or a parabola.

11°. Find the equation of a curve of the second order, tangent to the axis of x at the point (1, 0), and to the axis of y at the point (0, 2).

Solution.—The roots of equation (2), (last problem), must both be equal to 2, and the roots of equation (3) must both be equal to 1. Hence, we have,

$$d' = 2\sqrt{a'},$$

and,

$$e' = 2\sqrt{c'}.$$

Substituting these in (2) and (3), (last problem), and at the same time making $y=2$, and $x=1$, we have,

$$4a' + 4\sqrt{a'} = -1,$$

and,

$$c' + 2\sqrt{c'} = -1.$$

Whence,

$$\sqrt{a'} = -\frac{1}{2},$$

and,

$$\sqrt{c'} = -1.$$

Consequently,

$$a' = \frac{1}{4}, \quad c' = 1, \quad d' = -1, \quad \text{and} \quad e' = -2.$$

Substituting these values in equation (1), (last problem), multiplying by 4, and determining b' by the relation, $b'^2 = 4a'c' = 1$, we have,

$$y^2 + xy + 4x^2 - 4y - 8x + 4 = 0,$$

which is the required equation.

12°. Find the centre of the hyperbola,

$$y^2 + 4xy + 3x^2 - 6y - 5x - 3 = 0.$$

$$\textit{Ans.} \left(\frac{7}{2}, -4\right).$$

13°. Find the centre of the line,

$$y^2 - 2xy - x^2 - 4y - 8x + 8 = 0;$$

also determine the species of the curve.

Ans. Its centre is at the point $(-3, -1)$, *and the curve is an hyperbola.*

14°. Find the equation of the diameter of the last curve, that bisects all chords parallel to the axis of y.

Ans. $y = x + 2$.

15°. Find the slope of the principal axis of the curve in Problem 13°.

Ans. $\tan a = \tan 22° \; 30' = .414$.

16°. Find the equation of the ellipse,

$$3y^2 + 4xy + 2x^2 + y + 3x + \frac{5}{8} = 0 \quad . \quad . \quad (1)$$

referred to its centre, and axes parallel to the primitive ones.

Solution.—The required equation is of the form of (25), Article **89**. Comparing this with (22), we see that a, b, and c are the same as in the primitive equation, and that $-f'$, is what that absolute term in (22) becomes when m and n have the values shown in (23) and (24). To find the value of $-f'$, let us take the expression,

$$-f' = an^2 + bmn + cm^2 + dn + em + f \quad . \quad . \quad (2)$$

This can be written under the form,

$$-f' = \frac{1}{2}(2an + bm + d)n + \frac{1}{2}(2cm + bn + e)m$$
$$+ \frac{1}{2}(dn + em) + f.$$

Giving to m and n their values in (23) and (24), the first two terms reduce to 0; and after reduction, we have,

$$-f' = \frac{cd^2 + ae^2 - bde}{b^2 - 4ac} + f \quad . \quad . \quad . \quad . \quad (3)$$

In this particular case, we have,

$$a = 3, \quad b = 4, \quad c = 2, \quad d = 1, \quad e = 3, \quad \text{and} \quad f = \frac{5}{8}.$$

Hence, we have,

$$f' = \frac{3}{2};$$

and from equation (25) we have,

$$3y^2 + 4xy + 2x^2 = \frac{3}{2} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which is the required equation.

In like manner, any curve of the second order, that has a centre, may be transformed.

17°. Find the lengths of the semi-axes of the curve,

$$3y^2 + 4xy + 2x^2 = 3 \quad . \quad . \quad . \quad . \quad . \quad , \quad (1)$$

Solution.—If the co-ordinate axes be turned around the origin till the term bxy disappears, the new co-ordinate axes will coincide with the axes of the curve. In this case, the inclination of the new axis of x can be found from equation (3), Article **83**, and the values of a' and c' can be found from equations (5) and (6), Article **83**.

In this particular case, we have,

$$c' = \frac{1}{2}(5 + \sqrt{17}) = \frac{9.1}{2},$$

$$a' = \frac{1}{2}(5 - \sqrt{17}) = \frac{0.9}{2}.$$

Substituting these, together with the value of f', in equation (9), Article **87**, and multiplying by 20, we have,

$$91y^2 + 9x^2 = 60;$$

hence,

$$A = \sqrt{\frac{60}{9}} \quad \text{and} \quad B = \sqrt{\frac{60}{91}}. \quad \textit{Ans.}$$

PART II.

ANALYTICAL GEOMETRY OF THREE DIMENSIONS.

I. Preliminary Principles.

Rectangular Co-ordinates.

90. The relative positions of points in space may be determined by referring them to three planes, perpendicular to each other. These planes are called *co-ordinate planes.*

The *first* of these planes will be regarded as horizontal; the *second* will be taken perpendicular to the first, and in front of the observer, who is supposed to stand *on* the first plane; and the *third* will be taken perpendicular to both the others, and on the left of the observer.

The *first* of these planes is called the plane xy, the *second*, is called the plane xz, and the *third*, is called the plane yz.

These planes extend to an infinite distance in all directions, and divide all space into eight polyhedral angles.

The *first* of these angles lies above the plane xy, in front of the plane xz, and to the right of the plane yz; that is, it is the angle in which the observer is placed;

the *second* is to the left of the first; the *third* is behind the second; the *fourth* is to the right of the third; the *fifth* is below the first; the *sixth* is below the second; the *seventh* is below the third; and the *eighth* is below the fourth.

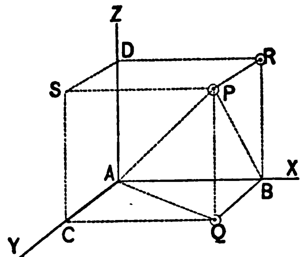

FIG. 52.

The co-ordinate planes intersect each other in three lines, AX, AY, and AZ, called co-ordinate axes; the line in which the planes xy and xz intersect, is called *the axis of x;* that in which the planes xy and yz intersect, is called *the axis of y;* and that in which the planes xz and yz intersect, is called *the axis of z.*

The *co-ordinates* of a point, are the distances of the point from the three co-ordinate planes. Let P be any point in space, and through it suppose three planes to be passed, parallel respectively to the three co-ordinate planes. These planes, together with the co-ordinate planes, will form a rectangular parallelopipedon, whose parallel edges are equal to each other. The distances, SP, RP, and QP (Fig. 52), are the co-ordinates of the point P; the first two, are sometimes called *abscissas,* and the third, is then called an *ordinate.*

The co-ordinates of a point are designated by the same letters as the axes to which they are parallel. Thus, SP, or its equal, AB, is denoted by x, RP, or its equal, AC, is denoted by y, and QP, or its equal, AD, is denoted by z.

Each co-ordinate is supposed to *spring from* the plane to which it is perpendicular, and to *terminate in* the point to which it belongs. The *abscissa* x, springs from the plane yz, and it is *positive* when estimated to the right, and *negative* when estimated to the left; the *abscissa* y, springs from the plane xz, and is *positive* when estimated to the front, and *negative* when estimated to the rear: the *ordinate* z, springs from the plane xy, and is *positive* when estimated upward, and *negative* when estimated downward.

Points are expressed by writing their co-ordinates in alphabetical order, within a parenthesis, and separating them from each other by commas; thus, (2, 3, 4), is the expression for *a point whose abscissa x is 2, whose abscissa y is 3, and whose ordinate z is 4.* The signs of the three co-ordinates of a point determine the angle in which the point is situated.

Polar Co-ordinates.

91. The point P will be determined in position when we know the distance AP, called *the radius-vector*, the angle XAQ, called the *azimuth*, and the angle QAP, called the *elevation* (Fig. 52). The point A is then called the *pole.*

Denoting the radius-vector of P by r, its azimuth by a, and its elevation by ϕ, we have from the figure,

$$x = r\cos\phi\cos a, \quad y = r\cos\phi\sin a, \quad \text{and} \quad z = r\sin\phi \quad . \ (1)$$

These equations enable us to pass from a system of rectangular to a system of polar co-ordinates, the pole being at the origin, and the plane from which azimuths are reckoned, being the plane xz.

Of Projections.

92. The *projection* of a point on a line, is the *foot* of a perpendicular from the point to the line. Thus, B, C, and D (Fig. 52), are the projections of the point P on the co-ordinate axes.

The *projection* of a point on a plane, is the foot of a perpendicular from the point to the plane. Thus, Q, R, and S are the projections of the point P on the co-ordinate planes.

In both cases the line that determines the projection of a point, is called the *projector* of that point.

The projection of one straight line on another is that portion of the latter which is intercepted between two planes passed through the extremities of the first, and perpendicular to the second. Thus, AB, AC, and AD, are the projections of the line AP, on the axes of co-ordinates.

It is obvious that the projections of the same line, on any two parallel lines, are equal, and that either projection is equal to the given line multiplied by the cosine of its inclination to the line on which the projection is made. It is to be understood that the angle between two lines that do not intersect, is equal to the angle between two lines drawn through any point, and parallel to the given lines. One straight line may be projected on another by projecting the extremities of the first upon the second.

The projection of a line on a plane is that line of the plane which joins the projections of the extremities of the given line. Thus, AQ is the projection of AP on

the plane xy. The plane through AP, perpendicular to the plane xy, is the projector of the line AP, and it is made up of the projectors of all the points of the line AP.

It is obvious that the projection of any line on a plane is equal to the given line multiplied by the cosine of its inclination to the plane. If the line is parallel to the plane, its projection is equal to the line itself.

The projection of a curve on a plane is made up of the projections of all its points. The projectors of the different points form a *cylindrical surface*, each projector being an *element* of the surface, and the projection being the *base* of the cylinder.

Proposition 44.—*To find a formula for the distance between any two points in space.*

93. Let it be required to find a formula for the distance between the points (x', y', z') and (x'', y'', z''). If three planes be passed through each point parallel to the co-ordinate planes, they will form by their intersections a rectangular parallelopipedon. The required distance is one of the diagonals of this parallelopipedon, and its *concurrent* edges, that is, the three edges that meet at one extremity of the diagonal, are respectively parallel to the co-ordinate axes. The edge parallel to the axis of x is equal to $x'' - x'$; that parallel to the axis of y is equal to $y'' - y'$; and that parallel to the axis of z is equal to $z'' - z'$. Denoting the required distance by d, and recollecting that the square of the diagonal of a rectangular parallelopipedon is equal to

the sum of the squares of three concurrent edges, we have,

$$d = \sqrt{(x'' - x')^2 + (y'' - y')^2 + (z'' - z')^2} \quad . \quad . \quad (1)$$

The quantities $x'' - x'$, $y'' - y'$, $z'' - z'$, are respectively equal to the projections of d on the co-ordinate axes x, y, and z. Hence, *the square of the length of any straight line in space, is equal to the sum of the squares of its projections on any three rectangular axes.*

Angles Between a Line and the Axes.

94. To find the relation between the cosines of the angles that a straight line makes with the co-ordinate axes, let us draw a line AP (Fig. 52) through the origin and parallel to the given line, and let us make its length, AP, equal to 1. The angles that AP makes with the co-ordinate axes are equal to the angles that the given line makes with those axes. Denoting these angles by X, Y, and Z, we have,

$$AB = AP\cos X, \quad AC = AP\cos Y, \quad \text{and} \quad AD = AP\cos Z.$$

But from the principle deduced in the last article, we have,

$$AP^2 = AB^2 + AC^2 + AD^2.$$

Substituting for AB, AC, and AD, their values, and remembering that $AP = 1$, we have,

$$1 = \cos^2 X + \cos^2 Y + \cos^2 Z \quad . \quad . \quad . \quad . \quad (1)$$

That is, *the sum of the squares of the cosines of the angles that any line makes with the axes of co-ordinates is equal to 1.*

II. Of the Straight Line.

Proposition 45.—*To find the equations of any straight line in space.*

95. Let BQ and CR be the projections of a straight line on the two planes xz and yz. Let the equations of these projections be

$$x = az + \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad [57]$$

$$y = bz + \beta \quad . \quad . \quad . \quad . \quad . \quad . \quad [58]$$

In which a and b are the tangents of the angles that the projections make with the axis of z, and α and β are the intercepts AB and AC.

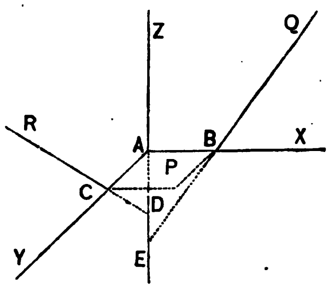

Fig. 53.

If two planes be passed through BQ and CR, the first perpendicular to the plane xz, and the second perpendicular to the plane yz, they will be the projectors of the given line; hence, the given line lies in both of these planes, and consequently, it is their line of intersection. Therefore, if any point (x, y, z) of the given line be projected on the plane xz, its projection will be on the line BQ; and since the co-ordinates x and z of this point in space are parallel and equal to the co-ordinates x and z of the projection of the point, it follows that equation [57] expresses the relation between the co-ordinates x and z of every point of the given line. In like manner, it may be shown that equation

[58] expresses the relation between the co-ordinates y and z of every point of the given line.

If we make [57] and [58] *simultaneous*, that is, if we make z the same in both, x, y, and z in the two equations will be the co-ordinates of every point of the line in space; but this line was any straight line; hence, *equations* (1) *and* (2) *are the equations of any straight line in space.*

Discussion of the Equations.

96. The ordinate of the point in which the line pierces the plane xy is equal to 0. Making $z = 0$ in [57] and [58], we find,

$$x = a \quad \text{and} \quad y = \beta.$$

Hence, the line pierces the plane xy in the point (a, β). To construct this point, we draw through B (Fig. 53) a line BP parallel to AY, and through C a line CP parallel to AX; the intersection, P, of these two lines is the required point.

If we make $y = 0$, we find,

$$x = \frac{ba - a\beta}{b}, \quad z = -\frac{\beta}{b};$$

hence, the line pierces the plane xz in the point,

$$\left(\frac{ba - a\beta}{b}, -\frac{\beta}{b}\right).$$

In like manner, we find that it pierces the plane yz in the point,

$$\left(\frac{a\beta - ba}{a}, -\frac{a}{a}\right).$$

If we combine equations [57] and [58], eliminating z, and transposing, we have,

$$y = \frac{b}{a}x - \left(\frac{b}{a}\alpha - \beta\right) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which expresses the relations between x and y for every point of the line; hence, it is the equation of the projection of the line on the plane xy.

Proposition 46.—*To find the equations of a straight line passing through a given point; also through two points.*

97. Let (x', y', z') be the given point; assume the equations of a straight line,

$$x = az + \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$y = bz + \beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the given point is on this line, its co-ordinates must satisfy its equations, giving,

$$x' = az' + \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$y' = bz' + \beta \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Conversely, if equations (3) and (4) are satisfied, the line (1), (2) must pass through the given point. Introducing these conditions by subtracting (3) from (1) and (4) from (2), we have,

$$x - x' = a(z - z') \quad . \quad . \quad . \quad . \quad . \quad [59]$$

$$y - y' = b(z - z') \quad . \quad . \quad . \quad . \quad [60]$$

which are the equations of a straight line through the given point.

The equations of condition that make the line just found, pass through the point (x'', y'', z'') are,

$$x'' - x' = a(z'' - z'), \quad \text{or,} \quad a = \frac{x'' - x'}{z'' - z'} \quad . \quad . \quad (5)$$

and,

$$y'' - y' = b(z'' - z'), \quad \text{or,} \quad b = \frac{y'' - y'}{z'' - z'} \quad . \quad . \quad (6)$$

Substituting these in [59] and [60], we have,

$$x - x' = \frac{x'' - x'}{z'' - z'}(z - z') \quad . \quad . \quad . \quad . \quad [59']$$

$$y - y' = \frac{y'' - y'}{z'' - z'}(z - z') \quad . \quad . \quad . \quad . \quad [60']$$

which are the equations of a straight line passing through the two points,

$$(x', y', z'), \quad \text{and} \quad (x'', y'', z'').$$

Proposition 47. — *To find a formula for the angle between two straight lines in space.*

98. The angle between two straight lines in space, whether they intersect or not, is equal to the angle formed by two straight lines drawn through a given point, and parallel to the given lines.

Let AP and AP′, each equal to 1, be drawn through the origin, and parallel to the given lines.

Denote the angle PAP′ by V, and the distance PP′ by d.

From the principle demonstrated in Davies' Mensuration, (Art. **97**), we have,

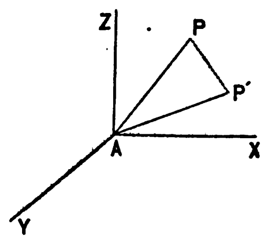

FIG. 54.

$$\cos V = \frac{\overline{AP}^2 + \overline{AP'}^2 - d^2}{2AP \times AP'} \quad . \quad . \quad . \quad . \quad (1)$$

Making AP and AP′ each equal to 1, we have,

$$\cos V = \frac{2 - d^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Denote the co-ordinates of P, by x', y', and z'; the co-ordinates of P′, by x'', y'', and z''; the angles between AP and the co-ordinate axes, by X, Y, and Z; and the angles between AP′ and the axes, by X′, Y′, and Z′

From the principle of projections (Art. **92**), we have,

$$x' = \text{AP} \cos X, \quad y' = \text{AP} \cos Y, \quad \text{and} \quad z' = \text{AP} \cos Z \;. \; (3)$$

$$x'' = \text{AP}' \cos X', \quad y'' = \text{AP}' \cos Y', \quad \text{and} \quad z'' = \text{AP}' \cos Z' \;. \; (4)$$

Making, $\text{AP} = 1$, $\text{AP}' = 1$, and substituting in formula (1), Article **93**, we have,

$$d^2 = (\cos X - \cos X')^2 + (\cos Y - \cos Y')^2 + (\cos Z - \cos Z')^2 \; (5)$$

Performing the operations indicated, and reducing by the relations, (Art. **94**),

$$\cos^2 X + \cos^2 Y + \cos^2 Z = 1,$$

and,

$$\cos^2 X' + \cos^2 Y' + \cos^2 Z' = 1,$$

we have,

$$d^2 = 2 - 2\,(\cos X \cos X' + \cos Y \cos Y' + \cos Z \cos Z') \;. \; (6)$$

Substituting this value of d^2, in equation (2), and reducing, we have,

$$\cos V = \cos X \cos X' + \cos Y \cos Y' + \cos Z \cos Z' \;. \; [61]$$

Hence, *the cosine of the angle between two straight*

lines in space, is equal to the sum of the rectangles of the cosines of the angles that the given lines make with the co-ordinate axes.

It is often more convenient to have the value of $\cos V$, in terms of the slopes of the projections of the lines on the two vertical planes. To deduce such an expression, let the equations of AP and AP′ be,

$$x = az, \quad y = bz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$x = a'z, \quad y = b'z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The equations of condition that place P on the line, (7), are,

$$x' = az', \quad \text{and} \quad y' = bz' \quad . \quad . \quad . \quad . \quad (9)$$

We have, also, the relation,

$$\overline{AP}^2 = x'^2 + y'^2 + z'^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Combining equations (9) and (10), we have,

$$\overline{AP}^2 = (a^2 + b^2 + 1)z'^2 \quad . \quad . \quad . \quad . \quad . \quad (11)$$

From equations (3), we have,

$$\cos X = \frac{x'}{AP}, \quad \cos Y = \frac{y'}{AP}, \quad \text{and} \quad \cos Z = \frac{z'}{AP} \quad . \quad (12)$$

Substituting for x', y', and AP, their values taken from (9) and (11), and reducing, we have,

$$\left.\begin{array}{l} \cos X = \dfrac{a}{\sqrt{1 + a^2 + b^2}}, \quad \cos Y = \dfrac{b}{\sqrt{1 + a^2 + b^2}}, \\ \text{and,} \\ \qquad \cos Z = \dfrac{1}{\sqrt{1 + a^2 + b^2}} \end{array}\right\} \quad . \quad (13)$$

In like manner, we can find,

$$\left.\begin{array}{l}\cos X' = \dfrac{a'}{\sqrt{1+a'^2+b'^2}}, \quad \cos Y' = \dfrac{b'}{\sqrt{1+a'^2+b'^2}}, \\ \text{and,} \\ \qquad \cos Z' = \dfrac{1}{\sqrt{1+a'^2+b'^2}}\end{array}\right\} \quad . \quad (14)$$

Substituting these values in equation [61], and reducing, we have,

$$\cos V = \frac{1 + aa' + bb'}{\sqrt{1 + a^2 + b^2}\sqrt{1 + a'^2 + b'^2}} \quad . \quad . \quad [62]$$

which is the required formula.

Discussion.

99. If the two lines are perpendicular to each other, $\cos V$ is equal to 0, that is,

$$1 + aa' + bb' = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Equation (1), is the equation of condition that makes two lines perpendicular to each other.

If one of the lines is given, we know a and b; there are then an infinite number of sets of values a' and b' that will satisfy equation (1); hence, there is an infinite number of lines that are perpendicular to a given straight line.

If the two lines are parallel, $\cos V$ is equal to 1; this requires that the numerator of [62] shall be equal to the denominator. But the only supposition that makes the numerator equal to the denominator, is,

$$a = a' \quad \text{and} \quad b = b' \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equations (2), are therefore the equations of condition that make two lines parallel to each other.

Since these equations only determine the direction of one line with respect to another, it follows that there may be an infinite number of lines, parallel to a given line.

PROBLEMS.

1°. Find the distance between the points $(3, 2, 1)$ and $(4, 5, 3)$.

Ans. $d = \sqrt{14}$.

2°. Find the distance between the points $(1, -2, -3)$ and $(4, -2, -1)$.

Ans. $d = \sqrt{13}$.

3°. Find the equations of a straight line through the point, $(2, 3, 4)$.

Ans. $\begin{cases} x - 2 = a(z - 4) \\ y - 3 = b(z - 4). \end{cases}$

4°. Find the equations of a straight line passing through the two points, $(3, 4, 2)$, and $(4, 1, 5)$.

Ans. $\begin{cases} 3x = z + 7 \\ 3y = -3z + 18. \end{cases}$

5°. Find the points in which the line last found pierces the co-ordinate planes.

Ans. $\left(\frac{7}{3}, 6, 0\right)$, $\left(\frac{13}{3}, 0, 6\right)$, and $\left(0, 13, -7\right)$.

6°. Find the equation of the projection of the line, found in Problem 4°, on the plane xy.

Ans. $3x = -y + 13$.

7°. Find the angle between the straight lines,

$$\begin{cases} x = 3z + 5 \\ y = 5z + 3, \end{cases}$$

and

$$\begin{cases} x = z + 1 \\ y = 2z. \end{cases}$$

Ans. 14° 58′.

8°. Find the equations of a line through the origin and perpendicular to both the lines of the last problem.

Solution.—The equations of any straight line through the origin are,

$$\left.\begin{aligned} x &= az \\ y &= bz \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The equation of condition that makes this line perpendicular to the first of the given lines is,

$$1 + 3a + 5b = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the equation of condition that makes it perpendicular to the second of the given lines is,

$$1 + a + 2b = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Combining (2) and (3), we find,

$$a = 3 \quad \text{and} \quad b = -2.$$

Substituting these in (1), we have,

$$x = 3z,$$
$$y = -2z;$$

which are the required equations.

9°. Find the equations of the line passing through the points $(2, 1, 0)$ and $(-3, 0, -1)$.

$$Ans. \quad \begin{cases} x = 5z + 2, \\ y = z + 1. \end{cases}$$

10°. Find the cosine of the angle between the lines,

$$\begin{cases} x = 2z + 1, \\ y = 2z + 2. \end{cases}$$

and,

$$\begin{cases} x = z + 5, \\ y = 4z + 1. \end{cases}$$

$$Ans. \quad \text{Cos V} = \frac{11}{9\sqrt{2}}.$$

III. Of the Plane.

Proposition 48. — *To find the equation of a plane.*

100. A plane may be generated by a straight line, revolving around one of its points, and continuing perpendicular to a given straight line. The revolving line is called the *generatrix*, and the fixed line is called the *directrix;* if the directrix pass through the origin of co-ordinates, it is called the *axis* of the plane.

Assume the equations of the axis,

$$x = az, \quad \text{and} \quad y = bz \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let (x', y', z') be any point in space, and let

$$x - x' = a'(z - z') \quad \text{and} \quad y - y' = b'(z - z') \quad .. \quad (2)$$

be the equations of a straight line passing through that point. These equations can be placed under the form,

$$a' = \frac{x - x'}{z - z'}, \quad \text{and} \quad b' = \frac{y - y'}{z - z'} \quad . \quad . \quad . \quad (3)$$

The condition that makes the line (3) perpendicular to the line (1) is, (Art. **99**),

$$1 + aa' + bb' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For one set of values of a' and b' that will satisfy equation (4), x, y, and z, in equations (3), will be the co-ordinates of every point of one position of the generatrix; hence, if we give to a' and b' every set of values they can have, and satisfy equation (4), x, y, and z, in equations (3), will become the co-ordinates of every point of the generatrix in every possible position, that is, they will be the co-ordinates of every point of the plane.

Hence, if we substitute in equation (4), the values of a' and b', taken from equations (3), the resulting equation will be the equation of the plane. Making this substitution, we have,

$$1 + a\frac{x - x'}{z - z'} + b\frac{y - y'}{z - z'} = 0 \quad . \quad . \quad . \quad . \quad (5)$$

Clearing of fractions and reducing,

$$z + ax + by - (z' + ax' + by') = 0.$$

Representing the quantity in the parenthesis by p, and transposing, we have,

$$z + ax + by = p \quad . \quad . \quad . \quad . \quad . \quad [63]$$

which is the required equation.

Discussion.

101. Equation [63], is of the first degree between three variables; in it, a, b, and p are arbitrary.

Conversely, every equation of the first degree between three variables is the equation of a plane; for, every equation of the first degree between three variables is a particular case of the general form.

$$Ax + By + Cz + D = 0 \quad . \quad . \quad . \quad . \quad (1)$$

Transposing, arranging, and dividing by C, we have,

$$z + \frac{A}{C}x + \frac{B}{C}y = -\frac{D}{C} \quad . \quad . \quad . \quad . \quad (2)$$

Equation (2) is of the same form as equation [63]; hence, it is the equation of a plane, that is, *every equation of the first degree, between three variables, is the equation of a plane.*

The *trace* of a plane on any of the co-ordinate planes, is its intersection with that co-ordinate plane.

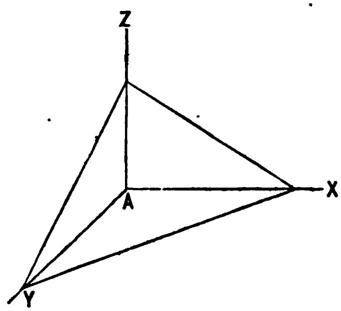

FIG. 55.

If we make $y = 0$ in equation [63], the resulting equation will express the relation between x and z, for every point of the plane whose abscissa y is equal to 0; that is, it will be the equation of the trace on the plane xz.

Making $y = 0$, in [63], and solving with respect to x, we have,

$$x = -\frac{1}{a}z + \frac{p}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which is the equation of the trace of the plane [63] on the plane xz.

In like manner, making $x = 0$, and solving with respect to y, we have,

$$y = -\frac{1}{b}z + \frac{p}{b} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which is the equation of the trace on the plane yz.

Making $z = 0$, and solving with respect to y, we have,

$$y = -\frac{a}{b}x + \frac{p}{b} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which is the equation of the trace on the plane xy.

Comparing equation (3) with the first of equations (1), in Article **100**, we see that the product of the coefficients of z, in the two equations, is equal to -1; this shows that the trace on the plane xz is perpendicular to the projection of the axis on the same plane.

Comparing (4) with the second of equations (1), Article **100**, we see, in like manner, that the trace on the plane yz is perpendicular to the projection of the axis on that plane.

Combining equations (1) of the last article, we find, for the projection of the axis on the plane xy,

$$y = \frac{b}{a}x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Comparing this with (5), we see that the trace on the plane xy is perpendicular to the projection of the axis on that plane.

If any line be drawn perpendicular to the plane, it

will be parallel to the axis of the plane, and its projections on the co-ordinate planes will, from the principle of projections, be parallel to the projections of the axis; hence, *if a line is perpendicular to a plane, the projections of the line are perpendicular to the traces of the plane.*

To find the points in which the plane intersects the axis of z, we make $x = 0$ and $y = 0$, which gives $z = p$; hence, the plane intersects the axis of z at a distance from the origin equal to p.

In like manner, we find the intercept on the axis of x to be $x = \frac{p}{a}$, and that on the axis of y to be $y = \frac{p}{b}$.

If we combine equations (1), of Article **100**, with the equation of the plane [63], we find,

$$\left.\begin{array}{c} z = \dfrac{p}{1 + a^2 + b^2}, \quad x = \dfrac{ap}{1 + a^2 + b^2}, \\ \text{and,} \qquad y = \dfrac{bp}{1 + a^2 + b^2} \end{array}\right\} \quad . \quad . \quad . \quad (7)$$

These values of z, x, and y, are the co-ordinates of the point in which the axis pierces the plane. Taking the square root of the sum of their squares, we find it equal to $\frac{p}{\sqrt{1 + a^2 + b^2}}$, which is the distance of the plane from the origin.

Proposition 49. — *To find a formula for the cosine of the angle between two planes.*

102. Assume the equations of the two planes,

$$z + ax + by = p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$z + a'x + b'y = p' \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The equations of the axes of these planes are,

$$x = az, \quad y = bz \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$x = a'z, \quad y = b'z \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Since the axes are, by definition, perpendicular to the planes, the angle between the axes is equal to the angle between the planes. But the cosine of the angle between the axes is given by formula [62]; hence, if we denote the angle between the two planes by V, we have,

$$\cos V = \frac{1 + aa' + bb'}{\sqrt{1 + a^2 + b^2}\sqrt{1 + a'^2 + b'^2}} \quad . \quad . \quad [64]$$

Two planes are perpendicular to each other when,

$$1 + aa' + bb' = 0.$$

And they are parallel to each other when,

$$a = a', \quad \text{and} \quad b = b'.$$

PROBLEMS.

1°. Find the equations of the traces of the plane,

$$z + 2x + 3y = 6.$$

On each of the co-ordinate planes,

$$Ans. \left\{ \begin{array}{ll} 2x + 3y = 6, & \text{on } xy. \\ z + 2x = 6, & \text{on } xz. \\ z + 3y = 6, & \text{on } yz. \end{array} \right.$$

2°. Find the intercepts of the same plane on the co-ordinate axes.

Ans. $x' = 3, \quad y' = 2, \quad z' = 6.$

3°. Find the ordinate of the point of the same plane whose projection on the plane, (xy), is $(3, 4)$.

Ans. $z' = -12$.

4°. Find the point in which the line,

$$\begin{cases} x = 2z + 3, \\ y = 3z - 2, \end{cases}$$

pierces the plane

$$z + 2x - y = 4.$$

Ans. $(-1, \quad -8, \quad -2)$.

5°. Find the line of intersection of the two planes,

$$z + 2x - y = 3,$$

$$z + x + 2y = 5.$$

$$Ans. \begin{cases} x = -\frac{3}{5}z + \frac{11}{5}, \\ y = -\frac{1}{5}z + \frac{7}{5}. \end{cases}$$

6°. Find the cosine of the angle between the same planes.

Ans. $\operatorname{Cos} V = \frac{1}{6}$.

7°. Find the equation of a plane in terms of its intercepts on the three co-ordinate axes.

Solution.—If we denote the intercepts of the plane,

$$z + ax + by = p \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

on the three axes taken in alphabetical order, by l, m, and n, we have,

$$l = \frac{p}{a}, \quad m = \frac{p}{b}, \quad \text{and} \quad n = \frac{p}{1}$$

whence,

$$a = \frac{p}{l}, \quad b = \frac{p}{m}, \quad \text{and} \quad 1 = \frac{p}{n}.$$

Substituting in equation (1), and reducing, we have,

$$\frac{z}{n} + \frac{x}{l} + \frac{y}{m} = 1,$$

which is the required equation.

8°. Find the equation of a plane that shall pass through the points (1, 2, 3), (2, 1, 2), and (3, 4, 3).

Ans. $z + \frac{1}{2}x - \frac{1}{2}y = \frac{5}{2}$.

9°. Find the equations of a straight line passing through the point $(-2, 3, 5)$, and perpendicular to the plane

$$z - 2x - 8y = -4.$$

Ans. $\left\{ \begin{array}{l} x = -2z + 8, \\ y = -8z + 43. \end{array} \right.$

10. Find the angle between the planes

$$z + \frac{5}{3}x - \frac{7}{3}y = -\frac{1}{3}.$$

$$z - \frac{2}{3}x - \frac{1}{3}y = 0.$$

Ans. 79° 52′.

11°. Find the distance from the point $(2, -3, 0)$ to the plane,

$$z - 8x - 9y = 2.$$

Ans. $-.75$.

12°. Find the polar equation of a plane.

Solution.—Assume the equation of a plane,

$$z + ax + by = p,$$

and in it substitute for x, y, and z their values taken from equation (1), (Art. **91**). The resulting equation,

$$r \sin \phi + ra \cos \phi \cos \alpha + rb \cos \phi \sin \alpha = p,$$

which may be written,

$$r = \frac{p}{\sin \phi + a \cos \phi \cos \alpha + b \cos \phi \sin \alpha},$$

is the required equation.

IV. Of Surfaces of Revolution.

Definitions.

103. A surface of revolution, is a surface that may be generated by any line, revolving about a straight line as an axis.

The revolving line, is called the *generatrix* of the surface. The fixed line, is called the *axis of revolution*, or simply *the axis*. A section of the surface by a plane passing through the axis, is called a *meridian-section*, and the plane itself, is called a *meridian-plane*. Any section of the surface by a plane perpendicular to the axis, is *a circle*.

It is obvious that all meridian-sections are equal to

each other. Hence, if any meridian-section be revolved about the axis, it will generate the surface.

Proposition 50. — *To find the general equation of a surface of revolution.*

104. Let the axis of Z be taken to coincide with the axis of revolution; let the generatrix be a meridian-curve cut from the surface by the plane xz, and let D be any point of the generatrix. Denote the co-ordinates of D by r and z; then will the equation of the generatrix be an equation expressing a relation between r and z. This relation may be expressed by the symbol,

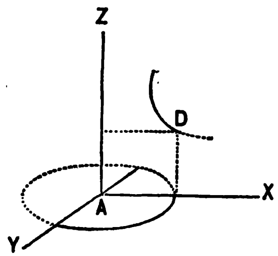

FIG. 56.

$$r = f(z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which is read, *r is a function of z;* that is, r depends on z for its value.

If the generatrix revolve about the axis of z, the point D will generate a circle parallel to the plane xy, and in every position of the point D, we shall have,

$$r^2 = x^2 + y^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now, for every value of r, we shall have a particular value of z, and this value of z, with the corresponding values of x and y, are the co-ordinates of every point of a horizontal circle of the surface. Consequently, if we suppose r to have every value that it can have and satisfy equation (1), the different values of z, with the correspond-

ing values of x and y, will be the co-ordinates of every point of every horizontal circle of the surface; that is, they will be the co-ordinates of every point of the surface. Hence, if we substitute the value of r, taken from (1), in equation (2), we have,

$$[f(z)]^2 = x^2 + y^2 \quad . \quad . \quad . \quad . \quad . \quad [65]$$

which is the general equation of a surface of revolution.

To apply this equation to any particular case, we take the equation of the curve as already deduced, changing x to r and y to z; the result is the equation of the meridian-curve in the plane xz; from it, we find the value of r^2 in terms of z; we then substitute this value in place of the first member of equation [65], and the result is the required equation.

In what follows, the same names are applied to surfaces as to the volumes they limit. Thus, the terms *sphere, spheroid*, and the like, are defined and used to designate surfaces, whereas in ordinary language they are employed to denote the volumes of which these surfaces are the limits.

Equation of a Sphere.

105. A *sphere* is a surface that may be generated by a circle, revolving about one of its diameters.

In this case, the equation of the generatrix is,

$$r^2 + z^2 = R^2 \quad . \quad . \quad (1)$$

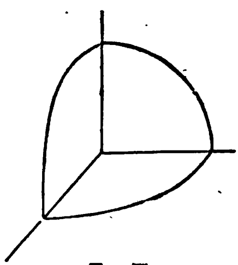

Fig. 57.

Finding the value of r^2, we have,

$$r^2 = R^2 - z^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting in [65], and reducing, we have,

$$x^2 + y^2 + z^2 = R^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad [66]$$

which is the equation of a sphere, whose radius is R.

Equation of an Oblate Spheroid.

106. An Oblate Spheroid is a surface that may be generated by an ellipse, revolving about its conjugate axis.

In this case the equation of the meridian curve is,

$$a^2z^2 + b^2r^2 = a^2b^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

whence,

$$r^2 = \frac{a^2}{b^2}(b^2 - z^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting in [65], aud reducing, we have,

$$b^2(x^2 + y^2) + a^2z^2 = a^2b^2 \quad . \quad . \quad . \quad . \quad [67]$$

which is the equation of the oblate spheroid.

Eqnation of a Prolate Spheroid.

107. A *prolate spheroid* is a surface that may be generated by an ellipse revolving about its transverse axis.

In this case the equation of meridian-curve is,

$$b^2z^2 + a^2r^2 = a^2b^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

whence,

$$r^2 = \frac{b^2}{a^2}(a^2 - z^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting in [65], and reducing, we have,

$$a^2(x^2 + y^2) + b^2z^2 = a^2b^2 \quad . \quad . \quad . \quad . \quad [68]$$

which is the equation of the prolate spheroid.

If we make $a = b$, in either of equations [67], or, [68], we have, after reduction,

$$x^2 + y^2 + z^2 = a^2 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which is the equation of a sphere whose radius is a.

Both the *oblate* and the *prolate spheroids* are *ellipsoids of revolution.*

Equation of an Hyperboloid of one Nappe.

108. A *nappe*, is a sheet, or branch of a surface.

An hyperboloid of one nappe, is a surface that may be generated by an hyperbola, revolving about its conjugate axis.

In this case, the equation of the meridian-curve is,

$$a^2z^2 - b^2r^2 = - a^2b^2 \quad . \quad . \quad . \quad . \quad (1)$$

whence,

$$r^2 = \frac{a^2}{b^2}(b^2 + z^2) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting in [65], and reducing, we have,

$$b^2(x^2 + y^2) - a^2z^2 = a^2b^2 \quad . \quad . \quad . \quad . \quad [69]$$

which is the equation of the hyperboloid of one nappe.

Equation of an Hyperboloid of two Nappes.

109. *An hyperboloid of two nappes*, is a surface that may be generated by an hyperbola revolving about its

transverse axis. Each branch of the curve generates a separate nappe, or branch of the surface.

In this case, the transverse axis of the curve is made to coincide with the axis of z, and the equation of the generatrix is,

$$a^2z^2 - b^2r^2 = a^2b^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

whence,

$$r^2 = \frac{a^2}{b^2}(z^2 - b^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting in [65], and reducing, we have,

$$b^2(x^2 + y^2) - a^2z^2 = -a^2b^2 \quad . \quad . \quad . \quad . \quad [70]$$

which is the equation of the hyperboloid of two nappes.

The generatrices in the last two cases, are conjugate hyperbolas, and the corresponding surfaces, are *conjugate hyperboloids.*

Equation of a Paraboloid of Revolution.

110. *A paraboloid of revolution,* is a surface that may be generated by a parabola revolving about its axis. Making the axis of the parabola coincide with the axis of z, we have for the equation of the generatrix,

$$r^2 = 2pz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Substituting in [65], and transposing, we have,

$$x^2 + y^2 - 2pz = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad [71]$$

which is the equation of a paraboloid of revolution.

Equation of a Cone.

111. A *cone* is a surface that may be generated by a straight line revolving about another straight line which it intersects.

The different positions of the generatrix are called *elements*, and the point of intersection is called the *vertex* of the cone.

In this case, the equation of the generatrix is,

$$z = ar + b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

whence,

$$r^2 = \frac{(z-b)^2}{a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting in [65], and reducing, we have,

$$a^2(x^2 + y^2) - (z - b)^2 = 0 \quad . \quad . \quad . \quad (3)$$

which is the equation of a cone.

In this equation, a is the slope of the generatrix. Denoting the inclination of the generatrix by α, we have,

$$a^2 = \tan^2 \alpha.$$

Substituting this in (3), we have,

$$(x^2 + y^2) \tan^2 \alpha - (z - b)^2 = 0 \quad . \quad . \quad . \quad [72]$$

which is the equation more commonly used.

In [72], b is the distance from the plane xy, to the vertex; the part of the cone that lies *below* the vertex, is called the *first nappe;* the part that lies *above* the vertex, the *second nappe;* both nappes extend to an infinite distance from the vertex that separates them.

The intersection of the cone with the plane xy is called the *base* of the cone.

Method of Discussion.

112. Many properties of the surfaces represented by

equations [66] to [72], may be determined by discussing the sections made by planes parallel to the planes of projection.

Planes parallel to either of the planes of projection are called *planes of parallel section.* The curves cut from a surface, by a system of such planes, are called *parallel sections* of the surface.

All parallel sections are projected on the corresponding co-ordinate plane, in their true dimensions. If a plane of parallel section does not cut the surface considered, the fact will be indicated by an imaginary equation for the projection of the curve. Conversely, an imaginary equation indicates that the plane does not intersect the surface.

Discussion of the Equations of the Spheroids.

113. If we make $z = q$ in equation [67], and reduce, we have,

$$x^2 + y^2 = \frac{a^2}{b^2}(b^2 - q^2) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Equation (1) is the equation of the projection, on the plane xy, of a section of the oblate spheroid made by a plane parallel to the plane xy, and at a distance from it equal to q. By giving to q every possible value, the equation may be made to represent every possible section parallel to the plane xy.

But equation (1) is the equation of a circle whose radius is equal to $\frac{a}{b}\sqrt{b^2 - q^2}$; the circle is real if $q^2 < b^2$; it reduces to a point if $q^2 = b^2$; and it is imaginary if $q^2 > b^2$.

Hence, every section of the surface parallel to the plane xy is a circle, and the surface is limited by two planes, parallel to the plane xy, and at distances from it equal to $+b$ and $-b$.

If we make, $x=m$, in [67], and reduce, we have, for the general equation of the projections of the sections parallel to the plane yz,

$$b^2y^2 + a^2z^2 = b^2(a^2 - m^2) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is the equation of an ellipse (Art. **88**). The ellipse is real if $m^2 < a^2$; it is a point if $m^2 = a^2$; and it is imaginary if $m^2 > a^2$, (Art. **87**).

Hence, all sections of the oblate spheroid, parallel to the plane yz, are ellipses, and the surface is limited by the parallel planes that are at a distance from the plane yz, equal to $+a$, and $-a$.

If we make $y=n$ in [67], and reduce, we have,

$$b^2x^2 + a^2z^2 = b^2(a^2 - n^2) \quad . \quad . \quad . \quad . \quad (3)$$

which is of the same form as (2), except that x and y have changed places.

Hence, the sections parallel to the plane xz, are ellipses, and the limiting planes are at distances from the plane xz equal to $+a$ and $-a$.

Equation [68] is the same as equation [67], except that a and b change places.

Hence, the preceding conclusions are applicable to the prolate spheroid, changing a into b, and b into a.

Discussion of the Equations of the Hyperboloids.

114. If we make $z=q$, in equations [69] and [70],

and reduce, we have, for the sections of the hyperboloids parallel to the plane xy,

$$x^2 + y^2 = \frac{a^2}{b^2}(q^2 + b^2) \quad . \quad . \quad . \quad . \quad (1)$$

$$x^2 + y^2 = \frac{a^2}{b^2}(q^2 - b^2) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Both (1) and (2) are the equations of circles; equation (1) is always real; equation (2) is imaginary if $q^2 < b^2$, but it is real if $q^2 > b^2$.

Hence, all the sections of both hyperboloids parallel to xy are circles. In the hyperboloid of one nappe, all the sections are real, and the surface has no limit in the direction of the axis of z. The hyperboloid of two nappes is limited by two planes parallel to the plane xy, and at distances from that plane equal to $+b$ and $-b$, no part of the surface lying between the limiting planes.

The smallest circle in the hyperboloid of one nappe is found by making $q = 0$. This circle is called the *circle of the gorge.*

If we make $x = m$, in [69] and [70], and reduce, we find for the general equations of the projections of sections parallel to the plane yz,

$$b^2y^2 - a^2z^2 = b^2(a^2 - m^2) \quad . \quad . \quad . \quad . \quad (3)$$

$$b^2y^2 - a^2z^2 = -b^2(a^2 + m^2) \quad . \quad . \quad . \quad (4)$$

Both (3) and (4) are equations of hyperbolas, (Art. 88). The transverse axis of the hyperbolas represented by equation (3) is parallel to the axis of z, if $m^2 < a^2$,

and it is parallel to the axis of y if $m^2 > a^2$, (Art. 88). If $m^2 = a^2$, equation (3) reduces to,

$$y = \pm \frac{a}{b} z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which is the equation of two straight lines intersecting at the origin. These lines are the asymptotes of the projections of all the sections parallel to the plane yz.

All the hyperbolas represented by (4) have their transverse axes parallel to the axis of z. If we make $y = n$, in [69] and [70], and reduce, we have for the equations of the projections of sections parallel to the plane xz,

$$b^2x^2 - a^2z^2 = b^2(a^2 - n^2) \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$b^2x^2 - a^2z^2 = -b^2(a^2 + n^2) \quad . \quad . \quad . \quad . \quad (7)$$

Equations (6) and (7) are of the same form as (3) and (4), except that x has taken the place of y, and n the place of m. Hence, the conclusions just deduced are applicable to the sections parallel to the plane xz, simply changing y into x, and m into n.

Discussion of the Equation of the Paraboloid.

115. Making $z = q$, in [71], and reducing, we have,

$$x^2 + y^2 = 2pq \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This equation shows that the sections parallel to the plane xy are all circles, real if p and q have the same sign, and imaginary if they have contrary signs.

Making $x = m$, and then making $y = n$, and re-

ducing, we have, for the projections of the sections parallel to the planes yz, and xz,

$$y^2 = 2pz - m^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$x^2 = 2pz - n^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equation (2) is of the same form as (3), except that x and y change places.

But these equations are equations of parabolas (Art. **88**). The parabolas lie above the plane xy when p is positive, and below it when p is negative (Art. **86**).

V. Of Conic Sections.

Method of Discussion.

116. In the preceding articles, we have only considered *parallel sections*. In discussing the sections of a cone, we shall pass a plane through the axis of y, making an arbitrary angle with the plane xy, and then find the equation of the resulting section referred to a pair of rectangular axes, in its own plane. Having found this equation, we shall suppose the cutting plane to assume all possible positions by turning it about the axis of y, and at the same time we shall suppose the cone to move *up* and *down* with respect to the plane xy, by changing the distance from the origin to the vertex. Since the cone is perfectly symmetrical with respect to the axis of z, we shall, in this manner, be able to determine the nature of every possible section of any given right cone.

General Equation of the Conic Section.

117. Let the plane YAD pass through the axis of y, and denote the angle XAD, which it makes with the plane xy, by u.

The plane xz divides the cone symmetrically, and the plane YAD is perpendicular to it; hence, the section of the cone made by the plane YAD is symmetrical with reference to the line KD.

Let AY and KD be taken as axes, and let the equation of the curve, APD, be referred to them.

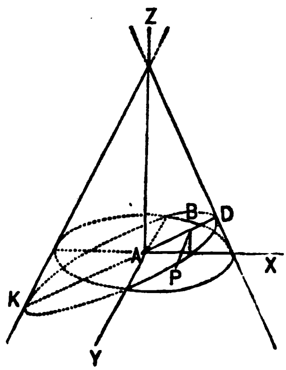

FIG. 58.

Take any point, P, of the curve, as the one that is projected on the plane xz at B, and denote its co-ordinates with respect to the original axes by x, y, and z, and with respect to the new axes, by x' and y'.

The distance of the point from the plane xz is the same as its distance from the line KD. The co-ordinates x', x, and z, form a right-angled triangle, having x' for its hypothenuse, and the angle at the base being u. Hence, we have the following relations:

$$x = x' \cos u, \quad y = y', \quad \text{and} \quad z = x' \sin u \quad . \quad . \quad (1)$$

But x, y, and z, are the co-ordinates of all the points of the conic surface that lie in the plane YAD, referred to AX, AY, and AZ as axes, and x' and y' are the co-ordinates of the same points referred to KD and AY as

axes. Hence, if we substitute for x, y, and z, in equation [72], their values taken from equations (1), we shall have the equation of the conic section referred to two rectangular axes in its own plane.

Making the substitutions, and replacing a by v, we have,

$$(x'^2 \cos^2 u + y'^2) \tan^2 v - (x' \sin u - b)^2 = 0 \quad . \quad . \quad (2)$$

Omitting the accents, performing indicated operations, and arranging, we have,

$$y^2 \tan^2 v + x^2 (\cos^2 u \tan^2 v - \sin^2 u) + 2b \sin ux - b^2 = 0 \, . \, . \, (3)$$

But, $\sin^2 u = \cos^2 u \tan^2 u$; substituting this in the second term of equation (3), and reducing, we have,

$$y^2 \tan^2 v + x^2 \cos^2 u (\tan^2 v - \tan^2 u) + 2b \sin ux - b^2 = 0 \, . \, . \, [73]$$

which is the general equation of a conic section.

Since equation [73] is of the second degree, we infer that all conic sections are curves of the second order (Art. **88**).

Discussion.

118. Comparing equation [73], with the general equation (1), Article **82**, we see that,

$$b = 0, \quad a = \tan^2 v, \quad \text{and} \quad c = \cos^2 u (\tan^2 v - \tan^2 u) \quad . \quad . \quad (1)$$

whence,

$$4ac = 4 \tan^2 v \cos^2 u (\tan^2 v - \tan^2 u) \quad . \quad . \quad . \quad (2)$$

1°. If $\tan^2 u = \tan^2 v$, we have, $4ac = 0$, or $b^2 = 4ac$;

hence, in this case, the conic section is a *parabola*, (Art. **88**).

2°. If $\tan^2 u < \tan^2 v$, we have, $4ac > 0$, or $b^2 < 4ac$; hence, in this case, the conic section is an *ellipse*.

3°. If $\tan^2 u > \tan^2 v$, we have, $4ac < 0$, or $b^2 > 4ac$; hence, in this case, the conic section is an *hyperbola*.

In the *first* case, the *cutting-plane* is parallel to one element of the cone; in the *second* case, the *cutting-plane makes a less angle* with the plane of the base of the cone, than that made by an element; in the *third* case, the *cutting-plane makes a greater angle* with the plane of the base, than that made by an element.

In the *first* case, all the elements of the cone are cut, except the one to which the cutting-plane is parallel, and the points of intersection are all on one nappe of the cone.

In the *second* case, all the elements, are cut by the cutting-plane, and in one nappe. In this case, the curve has but a single branch, which returns upon itself.

In the *third* case, all the elements are cut, except *two*. These two, lie in a plane through the vertex, parallel to the cutting-plane. Of the points of intersection, half lie on one nappe, and half on the other.

Hence, if a right cone with a circular base be intersected by a plane, the section will be a *parabola* when all the elements but one are cut, an *ellipse* when all the

elements are cut, and an *hyperbola*, when all the elements but two are cut.

For given values of v and u, the shape of the section remains unchanged, but its dimensions may be varied at pleasure, by giving different values to b, the ordinate of the vertex. Varying b alone, is equivalent to raising or depressing the cutting-plane, without changing its inclination, either with respect to the base or to the element.

If we make $b = 0$, the parabola reduces to a straight line, the ellipse reduces to a point, and the hyperbola to two straight lines, that intersect each other. Under this hypothesis the cutting-plane passes through the vertex.

A right cylinder, with a circular base, may be regarded as a limiting case of the cone. If a plane be passed parallel to the axis of such a surface, it will cut out two parallel lines, one line, or two imaginary parallels, according as its distance from the axis is *less than*, *equal to*, or *greater than*, the radius of the cylinder. These are the limiting cases of the parabola.

A plane passing through a point on the axis of z, and parallel to the plane xy, may be regarded as the other limiting case of a cone.

If a plane be passed parallel to this surface, its intersection with it, is an imaginary curve, which is one of the limiting curves of the ellipse. If a plane be passed parallel to the base of any other cone, its intersection with it is a circle, which is another limiting case of the

ellipse. If a plane be passed through the vertex, and parallel to the base, the circle reduces to a point.

Hence, every curve found by the discussion of the general equation of the second degree between two variables, is a conic section.